

# Exact renormalization group analysis in Hamiltonian theory: I. QED Hamiltonian on the light front

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## Abstract

The infinitesimal unitary transformation, introduced recently by F. Wegner, to bring the Hamiltonian to diagonal (or band diagonal) form, is applied to the Hamiltonian theory as an exact renormalization scheme. We consider QED on the light front to illustrate the method. The low-energy generated interaction, induced in the renormalized Hamiltonian to the order  $\alpha$ , is shown to be negative to insure, together with instantaneous term and perturbative photon exchange, the bound states for positronium. It is possible to perform the complete elimination of the  $ee\gamma$ -vertex in the instant form frame; this gives rise to the cutoff independent  $e\bar{e}$ -interaction governed by generated and instantaneous terms. The well known result for the singlet-triplet splitting  $\frac{7}{6}\alpha^2\text{Ryd}$  is recovered in the nonrelativistic limit as long as  $\lambda \ll m$ .

We examine the mass and wave function renormalization. The ultraviolet divergencies, associated with a large transverse momentum, are regularized by the regulator arising from the unitary transformation. The severe infrared divergencies are removed if all diagrams to the second order, arising from flow equations method and normal-ordering Hamiltonian, are taken into account. The electron (photon) mass in the renormalized Hamiltonian vary with UV cutoff in accordance with 1-loop renormalization group equations. This indicates an intimate connection between Wilson's renormalization and the flow equation method.

The advantages of the method in comparison with the naive renormalisation group approach are discussed.

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# 1 Introduction

We apply the method formulated by F. Wegner [1] of unitary transformation, to bring the Hamiltonian to diagonal (or near diagonal) form, to the problem of renormalization. We start with the regularized bare cutoff Hamiltonian, with the energy widths (the energy differences between the free states in a matrix element) restricted to be below the bare UV cutoff  $\Lambda$ . As renormalization step we perform the unitary transformation that removes the energy widths in the final Hamiltonian, to be below the final cutoff  $\lambda < \Lambda$ , resulting in the band-diagonal structure for the renormalized Hamiltonian. This corresponds to the Wilson renormalization procedure in the Lagrange theory of integrating out the fields of higher energies.

This unitary transformation can be written

$$U(\lambda, \Lambda) H_B(\Lambda, e_\Lambda, m_\Lambda, g(e_\Lambda, m_\Lambda)) U^\dagger(\lambda, \Lambda) = H_B(\lambda, e_\lambda, m_\lambda, g(e_\lambda, m_\lambda)) , \quad (1)$$

where  $e_\lambda, m_\lambda$  denote the set of finite number of independent coupling constants, in the case of QED - the  $ee\gamma$ -coupling constant and the fermion mass. Instead we generate in the renormalized Hamiltonian new effective interactions, denoted as  $g(e_\lambda, m_\lambda)$ , and corresponding to the operators of higher dimensions. They must be added to the bare cutoff theory as irrelevant operators, namely

$$g(e_\Lambda, m_\Lambda) = 0, \quad \Lambda \rightarrow \infty , \quad (2)$$

being manifest at the low-energy scale. The number of independent coupling constants of the theory is preserved, i.e.  $g \rightarrow 0$  when  $e \rightarrow 0, m \rightarrow 0$ .

By definition the renormalized Hamiltonian, see [2], in the literature also called the effective Hamiltonian, is

$$H_R = \lim_{\Lambda \rightarrow \infty} H_B . \quad (3)$$

For renormalizable theories there arise only a finite number of infinities, associated with the limit  $\Lambda \rightarrow \infty$ . They can be eliminated order by order in the running coupling  $e_\lambda$  by an appropriate choice of cutoff-dependent counterterms. The counterterms are chosen from the constraint on the renormalized Hamiltonian to run coherently with the cutoff.

This renormalization scheme, called similarity Hamiltonian approach, was presented by S. Glazek and K. Wilson [2] as an alternative to the traditional Lagrange method.

The similarity unitary transformation was used by these authors in light-front field theory to obtain the band-diagonal Hamiltonian with finite cutoff  $\lambda$ . For these purposes we exploit in this work the flow equations, formulated by Wegner [1], that perform infinitesimal unitary transformations in a continuous way.

We consider in this work two aspects of renormalized Hamiltonian. The first point is related with renormalization group analysis. The second one is connected with the low energy sector of the theory and bound state calculations.

In the next section we give the key ingredients of the flow equation method and discuss and its application to renormalization group analysis in Hamiltonian theory.

## 2 Flow equations

The flow equation is written

$$\frac{dH(l)}{dl} = [\eta(l), H(l)] \quad (4)$$

or for the matrix elements

$$\frac{dH_{ij}}{dl} = \eta_{ij}(E_j - E_i) + [\eta, H_I]_{ij} , \quad (5)$$

with the Hamiltonian  $H = H_0 + H_I$  divided into a free and an interacting part. Here  $\eta(l)$  is the generator of the unitary transformation, that determines in the leading order the evolution of the interacting part  $H_{I,ik}$ ,  $i \neq k$  or, vice versa, can be regarded as function of  $H_I$ , i.e.  $\eta(H_I)$ . The flow parameter  $l$  changes from  $l_\Lambda = 0$  as  $\Lambda \rightarrow \infty$ , that corresponds to the initial canonical Hamiltonian  $H_{can}$ , to some finite value  $l_\lambda$ , where the Hamiltonian has band diagonal form. The value  $l_\lambda = \infty$  corresponds to the diagonal (or block-diagonal, in the case when exact diagonalization is impossible) form of the Hamiltonian. The dimension of the flow parameter  $l$  is  $1/(\text{energy})^2$ .

In the problem of renormalization the physical sense of the UV cutoff can be assigned to the flow parameter  $l$ , explicitly it satisfies

$$l_\lambda = \frac{1}{\lambda^2} , \quad (6)$$

where  $\lambda$  is the UV cutoff. The unitary transformation removes the matrix elements between the free states with energy differences between the bare cutoff,  $\Lambda$ , and the final cutoff,  $\lambda$  ( $\lambda < \Lambda$ ), resulting in a band-diagonal form for  $H_\lambda$ .

We continue with the method of flow equations itself. The generator of the transformation, according to [1], is chosen

$$\eta_{ij} = [H_0, H]_{ij} = (E_i - E_j)H_{I,ij} \quad (7)$$

giving

$$\frac{dH_{ij}}{dl} = -(E_i - E_j)^2 H_{I,ij} + [\eta, H_I]_{ij} . \quad (8)$$

Generally the interacting part is associated with the coupling constant  $g$ . We proceed further in the frame of perturbative theory (PT) with respect to the interaction  $H_I$ , or therefore  $O(g)$ . From the equations above follows, that the generator  $\eta$  to the leading order  $O(H_I)$  generates the effective interaction to the order  $O(H_I^2)$ . The new term must be added to the initial Hamiltonian as irrelevant operator at  $l_\Lambda \rightarrow 0$  (i.e. vanishing at  $l = 0$ ,  $H_{gen}(l = 0) = 0$ ) but being nonzero for any finite value of  $l_\lambda$ . To make the effective interaction band-diagonal we need to introduce the generator to the next order  $O(H_I^2)$ , which in its turn generates the effective interaction to the third order  $O(H_I^3)$ . We truncate this series assuming the coupling constant to be small. Therefore for finite value of  $l_\lambda$  one has

$$\begin{aligned} H(l) &= H_0(l) + H^{(1)}(l) + H^{(2)}(l) + \dots \\ \eta(l) &= \eta^{(1)}(l) + \eta^{(2)}(l) + \dots , \end{aligned} \quad (9)$$

where the upper index denotes the order of PT, and  $H^{(1)}(l) = H_I(l)$ , with  $\lim_{\Lambda \rightarrow \infty} H_I(l_\Lambda = 0)$ , being the interaction present in the initial canonical Hamiltonian.

To the leading order of perturbative theory (PT)

$$\frac{dH_{ij}^{(1)}}{dl} = -(E_i - E_j)^2 H_{ij}^{(1)} . \quad (10)$$

Neglecting the dependence of the energies  $E_i$  on the flow parameter (being of higher order), we obtain the solution in the form

$$H_{ij}^{(1)}(l_\lambda) = H_{I,ij}(l_\lambda) = H_{I,ij}(l_\Lambda) \frac{f_{ij}(l_\lambda)}{f_{ij}(l_\Lambda)} , \quad (11)$$

where we have introduced

$$f_{ij}(l_\lambda) = \exp \left\{ -l_\lambda (E_i - E_j)^2 \right\} = \exp \left\{ - \left( \frac{E_i - E_j}{\lambda} \right)^2 \right\} . \quad (12)$$

The initial condition is defined at the bare cutoff  $\Lambda \rightarrow \infty$ , namely  $\lim_{\Lambda \rightarrow \infty} H_I(l_\Lambda)$  corresponds in the case of QED to the bare coupling constant  $e_0$ .

The structure of the expression for  $H_I(l_\lambda)$  is transparent. By the unitary transformation performed with the leading order generator  $\eta$ , we succeeded to eliminate far-off-diagonal elements from the Hamiltonian matrix, so that only matrix elements with energy differences  $|E_i - E_j| < \frac{1}{\sqrt{l}} = \lambda$  are present.

To the next to leading order  $O(H_I^2)$

$$\frac{dH_{ij}^{(2)}}{dl} = -(E_i - E_j)^2 H_{ij}^{(2)} + [\eta^{(1)}, H^{(1)}]_{ij} . \quad (13)$$

We introduce  $H_{ij}^{(2)}(l_\lambda) = f_{ij}(l_\lambda) \tilde{H}_{ij}^{(2)}(l_\lambda)$ , then the flow equation is rewritten (again neglecting the cutoff dependence of the energies  $E_i$ )

$$\frac{d\tilde{H}_{ij}^{(2)}}{dl} = \frac{1}{f_{ij}} [\eta^{(1)}, H^{(1)}]_{ij} . \quad (14)$$

This gives rise to the following solution

$$\tilde{H}_{ij}^{(2)}(l_\lambda) = \tilde{H}_{ij}^{(2)}(l_\Lambda) + \int_{l_\Lambda}^{l_\lambda} dl' \frac{1}{f_{ij}} [\eta^{(1)}, H^{(1)}]_{ij} . \quad (15)$$

We are not going to give further details. But it is worth to mention here, that in the case of QED the second order term  $H^{(2)}$  contributes (in different sectors) to the new generated interaction  $H^{(gen)}$  and to the electron (photon) self energy terms. This means that beginning from the second order the values  $\tilde{H}_{ij}^{(n)}(l_\lambda)$ , defined as  $H_{ij}^{(n)}(l_\lambda) = f_{ij}(l_\lambda) \tilde{H}_{ij}^{(n)}(l_\lambda)$ , contain two terms, the interaction part and counterterms. Generally it can be written

$$\tilde{H}^{(n)}(l_\lambda) = \tilde{H}_I^{(n)}(l_\lambda) + X^{(n)}(l_\lambda) . \quad (16)$$

The counterterms are determined from the coupling coherence condition, [2]. For the explicit treatment of the second order for QED, see later.

Generally, the system of self-consistent flow equations for the Hamiltonian  $H(l)$  and the generator of the unitary transformation  $\eta(l)$  can be written in matrix form as follows

$$\begin{aligned} \frac{dH_{ij}(l)}{dl} &= [\eta, H_I]_{ij} + \frac{d \ln f_{ij}}{dl} H_{ij} \\ \eta_{ij}(l) &= \frac{1}{E_i - E_j} \left( - \frac{d \ln f_{ij}}{dl} H_{ij} \right) , \end{aligned} \quad (17)$$

where the function  $f_{ij}$  determines how fast the non-diagonal part of the Hamiltonian matrix with  $|E_i - E_j| > \lambda$  vanishes with  $l$ ,  $f_{ij} = \exp(-l(E_i - E_j)^2)$ . Note, that this form of the function  $f_{ij}$  corresponds to the choice  $\eta = [H_0, H]$  for the generator. Other choices for the function  $f_{ij}(l)$  are possible (see Appendix A).

The structure of the flow equations, eq. (17), is transparent. The first term in eq. (17) is responsible for the renormalization of physical values (coupling constants, masses, wave

functions) and also for the structure of the new generated interactions to the higher order (with respect to the canonical interaction) being then present in the initial Hamiltonian; the second term in eq. (17) insures band diagonal structure of the renormalized Hamiltonian. As will be shown later, in the case of QED the physical masses run to the order  $g^2$ , while the coupling constant starts to run to the order  $g^3$  with the right renormalization group coefficients, i.e. with those obtained in standard perturbative theory.

The second equation for  $\eta$  is chosen in the form that the terms with small energy denominators ( $|E_i - E_j| \rightarrow 0$ ) effectively do not contribute in the renormalized Hamiltonian. This advantage, as compared with naive perturbative theory, enables to consider Hamiltonians with continuum spectrum. We mention that the problem of small energy denominators is solved also in other schemes (Appendix A).

### 3 Renormalized Hamiltonian $H_R$ to the second order

#### 3.1 Canonical light-front $QED_{3+1}$ Hamiltonian

We start with the canonical light-front QED Hamiltonian  $H_{can}$ , divided into free and interacting parts

$$P^- = H_{can} = \int dx^- d^2x^\perp (\mathcal{H}_0 + \mathcal{H}_I) . \quad (18)$$

In light-front gauge  $A^+ = A^0 + A^3 = 0$ , the constrained degrees of freedom  $A^-$  and  $\psi_-$  ( $\psi = \psi_+ + \psi_-$ ,  $\psi_\pm = \Lambda_\pm \psi$ ) can be removed explicitly; this gives the light-front gauge Hamiltonian defined through the independent physical fields  $A_\perp$  and  $\psi_+$  only [5]

$$\mathcal{H}_0 = \frac{1}{2}(\partial^i A^j)(\partial^i A^j) + \xi^+ \left( \frac{-\partial_\perp^2 + m^2}{i\partial^+} \right) \xi , \quad (19)$$

$$\mathcal{H}_I = \mathcal{H}_{ee\gamma} + \mathcal{H}_{ee\gamma\gamma} + \mathcal{H}_{eeee} \quad (20)$$

and

$$\mathcal{H}_{ee\gamma} = e\xi^+ \left[ -2\left(\frac{\partial^\perp}{\partial^+} \cdot A^\perp\right) + \sigma \cdot A^\perp \frac{\sigma \cdot \partial^\perp + m}{\partial^+} + \frac{\sigma \cdot \partial^\perp + m}{\partial^+} \sigma \cdot A^\perp \right] \xi , \quad (21)$$

$$\mathcal{H}_{ee\gamma\gamma} = -ie^2 \left[ \xi^+ \sigma \cdot A^\perp \frac{1}{\partial^+} (\sigma \cdot A^\perp \xi) \right] , \quad (22)$$

$$\mathcal{H}_{eeee} = 2e^2 \left[ \left( \frac{1}{\partial^+} (\xi^+ \xi) \right) \left( \frac{1}{\partial^+} (\xi^+ \xi) \right) \right] , \quad (23)$$

where  $\{\sigma^i\}$  are the standard  $2 \times 2$  Pauli matrices, and  $\partial^+ = 2\partial_- = 2\frac{\partial}{\partial x^-}$ . We have used the two-component representation for fermion fields introduced by Zhang and Harindranath [5]  $\psi_+ = \begin{pmatrix} \xi \\ 0 \end{pmatrix}$ . To simplify the calculations we rewrite all interactions through creation and annihilation operators. This turns out to be useful in the flow equations formalism, [3].

Following standard quantum field theory procedure we use the momentum-space representation for the field operators, [4] and [5],

$$\begin{aligned} \xi(x) &= \sum_s \chi_s \int \frac{dp^+ d^2p^\perp}{2(2\pi)^3} \theta(p^+) (b_{p,s} e^{-ipx} + d_{p,\bar{s}} e^{ipx}) \\ A^i(x) &= \sum_\lambda \int \frac{dq^+ d^2q^\perp}{2(2\pi)^3} \frac{\theta(q^+)}{\sqrt{q^+}} (\varepsilon_\lambda^i a_{q,\lambda} e^{-iqx} + h.c.) , \end{aligned} \quad (24)$$

where spinors are  $\chi_{1/2}^{tr} = (1, 0)$ ,  $\chi_{-1/2}^{tr} = (0, 1)$ , with  $\bar{s} = -s$  and polarization vectors  $\varepsilon_1^i = \frac{-1}{\sqrt{2}}(1, i)$ ,  $\varepsilon_{-1}^i = \frac{1}{\sqrt{2}}(1, -i)$ ; the integration running over the  $p^+ \geq 0$  only these states, that are allowed the light-front theory.

The corresponding (anti)commutation relations are

$$\begin{aligned} \{b_{p,s}, b_{p',s'}^+\} &= \{d_{p,s}, d_{p',s'}^+\} = \bar{\delta}_{p,p'} \delta_{ss'} \\ [a_{q,\lambda}, a_{q',\lambda'}^+] &= \bar{\delta}_{q,q'} \delta_{\lambda,\lambda'} \end{aligned} \quad , \quad (25)$$

where

$$\bar{\delta}_{p,p'} \equiv 2(2\pi)^3 \delta(p^+ - p'^+) \delta^{(2)}(p^\perp - p'^\perp) . \quad (26)$$

The light-front vacuum has trivial structure for both boson and fermion sectors, namely  $a_q|0\rangle = 0$ ;  $b_p|0\rangle = 0$ , simplifying the analytical calculations. The normalization of states is according to

$$\langle p_1, s_1 | p_2, s_2 \rangle = \bar{\delta}_{p_1,p_2} \delta_{s_1,s_2} , \quad (27)$$

where  $b_{p,s}^+|0\rangle = |p, s\rangle$ .

Making use of the field representation eq. (24), we have the following Fourier transformed for

the **free** Hamiltonian

$$H_0 = \sum_s \int \frac{dp^+ d^2 p^\perp}{2(2\pi)^3} \theta(p^+) \frac{p^{\perp 2} + m^2}{p^+} (b_{p,s}^+ b_{p,s} + d_{p,s}^+ d_{p,s}) + \sum_\lambda \int \frac{dq^+ d^2 q^\perp}{2(2\pi)^3} \theta(q^+) \frac{q^{\perp 2}}{q^+} a_{q,\lambda}^+ a_{q,\lambda} , \quad (28)$$

the leading order  $O(e)$  **ee** $\gamma$ -coupling

$$\begin{aligned} H_{ee\gamma} &= \sum_{\lambda_1 s_2} \int_{p_1 p_2 q} [g_{p_1 p_2 q}^*(l) \varepsilon_\lambda^i \tilde{a}_q + g_{p_1 p_2 q}(l) \varepsilon_\lambda^{i*} \tilde{a}_{-q}^+] (\tilde{b}_{p_2}^+ \tilde{b}_{p_1} + \tilde{b}_{p_2}^+ \tilde{d}_{-p_1}^+ + \tilde{d}_{-p_2} \tilde{b}_{p_1} + \tilde{d}_{-p_2} \tilde{d}_{-p_1}^+) \\ &\quad \times \chi_{s_2}^+ \Gamma_l^i(p_1, p_2, -q) \chi_{s_1} \bar{\delta}_{q, p_2 - p_1} , \end{aligned} \quad (29)$$

where

$$\Gamma_l^i(p_1, p_2, q) = 2 \frac{q^i}{q^+} - \frac{\sigma \cdot p_2^\perp - im}{p_2^+} \sigma^i - \sigma^i \frac{\sigma \cdot p_1^\perp + im}{p_1^+} , \quad (30)$$

where the mass is l-dependent. Further we have for the **instantaneous** interactions of the order  $O(e^2)$

$$\begin{aligned} H_{eeee}^{inst} &= \sum_{s_1 s_2 s_3 s_4} \int_{p_1 p_2 p_3 p_4} g_{p_1 p_2 p_3 p_4}^{eeee}(l) (\tilde{b}_{p_3}^+ + \tilde{d}_{-p_3}) (\tilde{b}_{p_4}^+ + \tilde{d}_{-p_4}) (\tilde{b}_{p_1} + \tilde{d}_{-p_1}^+) (\tilde{b}_{p_2} + \tilde{d}_{-p_2}^+) \\ &\quad \times \chi_{s_3}^+ \chi_{s_4}^+ \frac{4}{(p_1^+ - p_3^+)^2} \chi_{s_1} \chi_{s_2} \bar{\delta}_{p_3 + p_4, p_1 + p_2} \end{aligned} \quad (31)$$

and

$$\begin{aligned} H_{ee\gamma\gamma}^{inst} &= \sum_{s_1 s_2 \lambda_1 \lambda_2} \int_{p_1 p_2 q_1 q_2} g_{p_1 p_2 q_1 q_2}^{ee\gamma\gamma}(l) (\varepsilon_{\lambda_1}^{i*} \tilde{a}_{q_1}^+ + \varepsilon_{\lambda_1}^i \tilde{a}_{-q_1}) (\varepsilon_{\lambda_2}^j \tilde{a}_{q_2} + \varepsilon_{\lambda_2}^{j*} \tilde{a}_{-q_2}^+) (\tilde{b}_{p_2}^+ + \tilde{d}_{-p_2}) (\tilde{b}_{p_1} + \tilde{d}_{-p_1}^+) \\ &\quad \times \chi_{s_2}^+ \frac{\sigma^j \sigma^i}{(p_1^+ - q_1^+)} \chi_{s_1} \bar{\delta}_{p_1 + q_2, q_1 + p_2} ; \end{aligned} \quad (32)$$

here

$$\begin{aligned} \tilde{a}_q &\equiv a_{q,\lambda} \frac{\theta(q^+)}{\sqrt{q^+}}, & \left[ \tilde{a}_{-q} &\equiv a_{-q,\lambda} \frac{\theta(-q^+)}{\sqrt{-q^+}} \right] , \\ \tilde{b}_p &\equiv b_{p,s} \theta(p^+), & \tilde{d}_p &\equiv d_{p,\bar{s}} \theta(p^+) , \end{aligned} \quad (33)$$

and the  $\bar{\delta}$  stands for the function defined in eq. (26), the short notation for the integral we understand as

$$\int_p \equiv \int \frac{dp^+ d^2 p^\perp}{2(2\pi)^3} . \quad (34)$$

In the formulas above we write explicitly the momentum dependence of the coupling constants as long as  $l \neq 0$ . The initial conditions for the couplings are defined at the value of the bare cutoff  $\Lambda \rightarrow \infty$ , namely

$$\lim_{\Lambda \rightarrow \infty} g^{ee\gamma}(l_\Lambda) = e_0 \quad (35)$$

and for both instantaneous interaction couplings

$$\lim_{\Lambda \rightarrow \infty} g^{inst}(l_\Lambda) = e_0^2 ; \quad (36)$$

these correspond to the couplings of the canonical theory.

## 3.2 The flow equations in $|e\bar{e}\rangle$ -sector

### 3.2.1 Generated interaction

Following the procedure outlined in the second section, the leading order generator of the unitary transformation is

$$\begin{aligned} \eta^{(1)}(l) = & \sum_{\lambda s_1 s_2} \int_{p_1 p_2 q} (\eta_{p_i p_f}^*(l) \varepsilon_\lambda^i \tilde{a}_q + \eta_{p_i p_f}(l) \varepsilon_\lambda^{i*} \tilde{a}_{-q}^+) (\tilde{b}_{p_2}^+ \tilde{b}_{p_1} + \tilde{b}_{p_2}^+ \tilde{d}_{-p_1}^+ + \tilde{d}_{-p_2} \tilde{b}_{p_1} + \tilde{d}_{-p_2} \tilde{d}_{-p_1}^+) \\ & \times \chi_{s_2}^+ \Gamma_l^i(p_1, p_2, -q) \chi_{s_1} \bar{\delta}_{q, p_2 - p_1} , \end{aligned} \quad (37)$$

where  $p_i$  and  $p_f$  stand for the set of initial and final momenta, respectively, and

$$\eta_{p_i p_f}(l) = -\Delta_{p_i p_f} g_{p_i p_f} = \frac{1}{\Delta_{p_i p_f}} \cdot \frac{dg_{p_i p_f}}{dl} . \quad (38)$$

Further we calculate the bound states of positronium. In what follows consider in  $|e\bar{e}\rangle$  sector

the **generated interaction** to the first nonvanishing order

$$H_{e\bar{e}e\bar{e}}^{gen} = \sum_{s_1 \bar{s}_2 s_3 \bar{s}_4} \int_{p_1 p_2 p_3 p_4} V_{p_i p_f}^{gen}(l) b_{p_3}^+ d_{p_4}^+ d_{p_2} b_{p_1} \chi_{s_3}^+ \chi_{\bar{s}_4}^+ \chi_{\bar{s}_2} \chi_{s_1} \bar{\delta}_{p_1 + p_2, p_3 + p_4} , \quad (39)$$

with the initial condition  $\lim_{\Lambda \rightarrow \infty} V_{p_i p_f}^{gen}(l_\Lambda) = 0$ ,

and the **instantaneous interaction**

$$H_{e\bar{e}e\bar{e}}^{inst} = \sum_{s_1 \bar{s}_2 s_3 \bar{s}_4} \int_{p_1 p_2 p_3 p_4} V_{p_i p_f}^{inst}(l) b_{p_3}^+ d_{p_4}^+ d_{p_2} b_{p_1} \chi_{s_3}^+ \chi_{\bar{s}_4}^+ \chi_{\bar{s}_2} \chi_{s_1} \bar{\delta}_{p_1 + p_2, p_3 + p_4} , \quad (40)$$

where

$$V_{p_i p_f}^{inst}(l) = g_{p_i p_f}^{inst}(l) \frac{4}{(p_1^+ - p_3^+)^2}(l) . \quad (41)$$

The ordering of the field operators in both interactions is important because it defines the kind of interaction, attractive or repulsive; the ordering given satisfies the standard Feynmann rule prescription in the  $|e\bar{e}\rangle$  sector.

We neglect the  $l$  dependence of momenta in the interaction, which enables us to write the flow equations for the corresponding couplings.

The flow equations to the order  $O(e^2)$

$$\begin{aligned}
\frac{dg_{p_i p_f}(l)}{dl} &= -\Delta_{p_i p_f}^2 g_{p_i p_f}(l) \\
\frac{dg_{p_i p_f}^{inst}(l)}{dl} &= -\Delta_{p_i p_f}^2 g_{p_i p_f}^{inst}(l) \\
\frac{dV_{p_i p_f}^{gen}(l)}{dl} &= < [\eta^{(1)}(l), H_{ee\gamma}] >_{|e\bar{e}} - \Delta_{p_i p_f}^2 V_{p_i p_f}^{gen}(l) ,
\end{aligned} \tag{42}$$

where

$$\Delta_{p_i p_f} = \sum p_i^- - \sum p_f^- \tag{43}$$

and the light-front fermion energie is  $p^- = \frac{p^{\perp 2} + m^2}{p^+}$ , the photon one  $q^- = \frac{q^{\perp 2}}{q^+}$ . The matrix element  $< [\eta^{(1)}(l), H_{ee\gamma}] >_{|e\bar{e}}$  is understood as the corresponding commutator between the free electron-positron states, namely  $< p_3 s_3, p_4 \bar{s}_4 | \dots | p_1 s_1, p_2 \bar{s}_2 >$ .

Neglecting the dependence of the momenta (the fermion mass) on the flow parameter  $l$ , the solution reads

$$\begin{aligned}
g_{p_i p_f}(l) &= f_{p_i p_f} \cdot e_0 + O(e^3) \\
g_{p_i p_f}^{inst}(l) &= f_{p_i p_f} \cdot e_0^2 + O(e^4) \\
V_{p_i p_f}^{gen}(l) &= f_{p_i p_f} \cdot \int_0^l dl' \frac{1}{f_{p_i p_f}(l')} < [\eta^{(1)}(l'), H_{ee\gamma}(l')] >_{|e\bar{e}} ,
\end{aligned} \tag{44}$$

where the subscript indicates, that the commutator is considered in the electron-positron sector. As was discussed in the previous section, the function  $f_{p_i p_f}$  eq. (12) insures band-diagonal structure for the renormalized interaction.

The matrix element of the commutator  $[\eta^{(1)}, H_{ee\gamma}]$  in the exchange and annihilation channels is (Appendix B)

$$< [\eta^{(1)}, H_{ee\gamma}] > / \delta_{p_1+p_2, p_3+p_4} = \begin{cases} M_{2ii}^{ex} \frac{1}{(p_1^+ - p_3^+)} (\eta_{p_1, p_3} g_{p_4, p_2} + \eta_{p_4, p_2} g_{p_1, p_3}) , \\ -M_{2ii}^{an} \frac{1}{(p_1^+ + p_2^+)} (\eta_{p_1, -p_2} g_{p_4, -p_3} + \eta_{p_4, -p_3} g_{p_1, -p_2}) , \end{cases} \tag{45}$$

where

$$\begin{aligned}
\eta_{p_1, p_2} &= e_0 \cdot \frac{1}{\Delta_{p_1 p_2}} \frac{df_{p_1, p_2}}{dl} \\
g_{p_1, p_2} &= e_0 \cdot f_{p_1, p_2}
\end{aligned} \tag{46}$$

and  $\Delta_{p_1, p_2} = p_1^- - p_2^- - (p_1 - p_2)^-$ . In what follows we drop the index 0 at the coupling. The matrix elements  $M_{2ii}$  between the corresponding spinors in both channels

$$\begin{aligned}
M_{2ij}^{(ex)} &= [\chi_{s_3}^+ \Gamma_l^i(p_1, p_3, p_1 - p_3) \chi_{s_1}] [\chi_{\bar{s}_2}^+ \Gamma_l^j(-p_4, -p_2, -(p_1 - p_3)) \chi_{\bar{s}_4}] \\
M_{2ij}^{(an)} &= [\chi_{s_3}^+ \Gamma_l^i(-p_4, p_3, -(p_1 + p_2)) \chi_{\bar{s}_4}] [\chi_{\bar{s}_2}^+ \Gamma_l^j(p_1, -p_2, p_1 + p_2) \chi_{s_1}]
\end{aligned} \tag{47}$$

determine the spin structure of the effective interaction. It is calculated explicitly in light-front frame in Appendix B.



The form of the second order renormalized interactions eq. (45), expressed through the  $f$ -function (that defines the behaviour of the first order coupling), is universal for the different renormalization schemes (see Appendix). (This is true up to some factor in the integral for the generated interaction). Specifying the  $f$  function we obtain the explicit form of the renormalized interactions for the different unitary transformations. In Appendix C we compare the results for the second order generated interaction in two renormalization schemes.

Here we choose the  $f$ -function as

$$f_{p_i, p_f} = \exp(-l \Delta_{p_i, p_f}^2) . \quad (48)$$

Then, neglecting again the dependence of the momenta on the flow parameter  $l$ , we have for the generated interaction in both channels

$$\begin{aligned} V^{(ex)}(l) &= -e^2 M_{2ii}^{(ex)} \frac{1}{(p_1^+ - p_3^+)} (\Delta_{p_1, p_3} + \Delta_{p_4, p_2}) f_{p_i p_f} \cdot \int_0^l dl' \frac{f_{p_1, p_3}(l') f_{p_4, p_2}(l')}{f_{p_i p_f}(l')} \\ V^{(an)}(l) &= e^2 M_{2ii}^{(an)} \frac{1}{(p_1^+ + p_2^+)} (\Delta_{p_1, -p_2} + \Delta_{p_4, -p_3}) f_{p_i p_f} \cdot \int_0^l dl' \frac{f_{p_1, -p_2}(l') f_{p_4, -p_3}(l')}{f_{p_i p_f}(l')} . \end{aligned} \quad (49)$$

This gives rise to

$$\begin{aligned} V_{gen}^{(ex)}(l) &= -e^2 M_{2ii}^{(ex)} \frac{1}{(p_1^+ - p_3^+)} \frac{1}{2} \left( \frac{1}{\Delta_{p_1, p_3}} + \frac{1}{\Delta_{p_4, p_2}} \right) \cdot \left( 1 - e^{-l \cdot 2 \Delta_{p_1, p_3} \Delta_{p_4, p_2}} \right) \cdot e^{-l \Delta_{p_i p_f}^2} \\ V_{gen}^{(an)}(l) &= e^2 M_{2ii}^{(an)} \frac{1}{(p_1^+ + p_2^+)} \frac{1}{2} \left( \frac{1}{\Delta_{p_1, -p_2}} + \frac{1}{\Delta_{p_4, -p_3}} \right) \cdot \left( 1 - e^{-l \cdot 2 \Delta_{p_1, -p_2} \Delta_{p_4, -p_3}} \right) \cdot e^{-l \Delta_{p_i p_f}^2} , \end{aligned} \quad (50)$$

where

$$\Delta_{p_i p_f} \equiv p_1^- + p_2^- - p_3^- - p_4^- = \Delta_{p_1, p_3} - \Delta_{p_4, p_2} = \Delta_{p_1, -p_2} - \Delta_{p_4, -p_3} \quad (51)$$

due to momentum conservation in '+' and 'transversal' directions.

For energy conserving processes, i.e. when  $\Delta_{p_i p_f} = 0$ , the unity in eq. (51) gives the result of the standard renormalization procedure. This interaction contains divergencies in the form of small energy denominator, as is general for perturbative approach. This problem is cured in the method of flow equations (and also by similarity transformations) by the proper choice of the generator  $\eta$  (Appendix C). The divergencies in eq. (51) effectively are cancelled by the exponential factor in the bracket  $(1 - \exp)$  as long as the leading order  $ee\gamma$  coupling is not completely eliminated (i.e. for the finite cutoff  $l_\lambda$ ).

We rewrite the renormalized to the second order **O(e<sup>2</sup>) effective interaction**, eq. (51), as

$$\begin{aligned} V_{gen, \lambda}^{(ex)} &= -e^2 N_{1, \lambda} \frac{1}{2} \left( \frac{1}{\tilde{\Delta}_1} + \frac{1}{\tilde{\Delta}_2} \right) \cdot \left( 1 - e^{-2 \frac{\tilde{\Delta}_1}{\lambda^2} \frac{\tilde{\Delta}_2}{\lambda^2}} \right) \cdot e^{-\left( \frac{M_0^2 - M_0'^2}{\lambda^2} \right)^2} \\ V_{gen, \lambda}^{(an)} &= e^2 N_{2, \lambda} \frac{1}{2} \left( \frac{1}{M_0^2} + \frac{1}{M_0'^2} \right) \cdot \left( 1 - e^{-2 \frac{M_0^2}{\lambda^2} \frac{M_0'^2}{\lambda^2}} \right) \cdot e^{-\left( \frac{M_0^2 - M_0'^2}{\lambda^2} \right)^2} , \end{aligned} \quad (52)$$

where we have introduced

$$P^{+2} M_{2ii, \lambda}^{(ex)} = -N_1 \quad ; \quad P^{+2} M_{2ii, \lambda}^{(an)} = N_2$$

$$\begin{aligned}
\Delta_{p_1 p_3} &= \frac{\Delta_1}{P^+} = \frac{\tilde{\Delta}_1}{(x'-x)P^+} \quad ; \quad \Delta_{p_4 p_2} = \frac{\Delta_2}{P^+} = \frac{\tilde{\Delta}_2}{(x'-x)P^+} ; \\
\Delta_{p_1, -p_2} &= \frac{M_0^2}{P^+} \quad ; \quad \Delta_{p_4, -p_3} = \frac{M_0'^2}{P^+}
\end{aligned} \tag{53}$$

(see Appendix B for the explicit definition of these quantities in the light-front frame).

The expression eq. (53) is written for the rescaled value of the potential  $V \rightarrow P^{+2}V$ , and the cutoff is defined in units of the total momentum  $P^+$ , i.e.  $\lambda \rightarrow \frac{\lambda^2}{P^+}$ , with  $l = 1/\lambda^2$ . The spin structure of the interaction is carried by the matrix elements  $M_{2ii}$ , defined in Appendix B.

We summarize the **instantaneous interaction** in both channels to the order  $\mathbf{O}(\mathbf{e}^2)$ , cf. fig. (1),

$$V_{inst}^{(ex)} = -\frac{4e^2}{(p_1^+ - p_3^+)^2} \delta_{s_1 s_3} \delta_{s_2 s_4} \exp(-l \Delta_{p_i p_f}^2) \tag{54}$$

$$V_{inst}^{(ex)} = \frac{4e^2}{(p_1^+ + p_2^+)^2} \delta_{s_1 \bar{s}_2} \delta_{s_3 \bar{s}_4} \exp(-l \Delta_{p_i p_f}^2) ,$$

where we have used  $\chi_{s_3}^+ \chi_{\bar{s}_2}^+ \mathbb{1} \chi_{s_1} \chi_{\bar{s}_4} = \delta_{s_1 s_3} \delta_{s_2 s_4} + \delta_{s_1 \bar{s}_2} \delta_{s_3 \bar{s}_4}$ . For the rescaled potential in the light-front frame (fig. (4) and Appendix B eq. (194) and following) we thus have

$$\begin{aligned}
V_{inst, \lambda}^{(ex)} &= -\frac{4e^2}{(x - x')^2} \delta_{s_1 s_3} \delta_{s_2 s_4} \exp \left\{ - \left( \frac{M_0^2 - M_0'^2}{\lambda^2} \right)^2 \right\} \\
V_{inst, \lambda}^{(an)} &= 4e^2 \delta_{s_1 \bar{s}_2} \delta_{s_3 \bar{s}_4} \exp \left\{ - \left( \frac{M_0^2 - M_0'^2}{\lambda^2} \right)^2 \right\} ,
\end{aligned} \tag{55}$$

where the notations of eqsgil4agi16 are implied.

### 3.2.2 Renormalization issues

As was discussed above after eq. (17) the commutator  $[\eta^{(1)}, H_{ee\gamma}]$  also contributes to the self-energy term, giving rise to the renormalization of fermion and photon masses to the second order. The flow equation for the electron (photon) light-cone energy  $p^-$  is

$$\frac{dp^-}{dl} = \langle [\eta^{(1)}, H_{ee\gamma}] \rangle_{selfenergy} , \tag{56}$$

where the matrix element is calculated between the dressed single electron (photon) states  $\langle p', s' | \dots | p, s \rangle$ . We drop the finite part and define  $\delta p_\lambda^- = p^-(l_\lambda) - \langle |H_0| \rangle$ . Integration over the finite range gives

$$\delta p_\lambda^- - \delta p_\Lambda^- = \int_{l_\Lambda}^{l_\lambda} \langle [\eta^{(1)}, H_{ee\gamma}] \rangle_{selfenergy} dl' = -\frac{(\delta \Sigma_\lambda(p) - \delta \Sigma_\Lambda(p))}{p^+} , \tag{57}$$

that defines the cutoff dependent self energy  $\delta \Sigma_\lambda(p)$ . The mass correction and wave function renormalization constant correspondingly are given, cf. [5], as

$$\begin{aligned}
\delta m_\lambda^2 &= p^+ \delta p^- \Big|_{p^2=m^2} = -\delta \Sigma_\lambda(m^2) \\
Z_2 &= 1 + \frac{\partial \delta p^-}{\partial p^-} \Big|_{p^2=m^2} .
\end{aligned} \tag{58}$$

The on-mass-shell condition is defined through the mass  $m$  in the free Hamiltonian  $H_0$ .

We show further, that to the second order  $O(e^2)$  the electron and photon masses and corresponding wave function renormalization constants in the renormalized Hamiltonian vary in accordance with the result of 1-loop renormalization group equations. This can serve as evidence for the equivalence of the herediscussed method of flow equations and Wilson's renormalization scheme. Therefore we have rewritten the mass correction  $\delta m_\lambda^2$  through the self energy term, arising in 1-loop calculations of ordinary perturbative theory. The negative overall sign stems from our definition of the flow parameter, namely for  $\Delta l > 0$  we are lowering the cutoff  $dl = -\frac{2}{\lambda^3}d\lambda$ .

We start with the bare cutoff mass  $m_\Lambda^2 = m^2 + \delta M_\Lambda^{(2)}$ , where  $\delta M_\Lambda^{(2)}$  is the second order mass counterterm. According to eq. (57) the electron (photon) mass runs

$$m_\lambda^2 = m_\Lambda^2 - [\delta \Sigma_\lambda(m^2) - \delta \Sigma_\Lambda(m^2)] \quad (59)$$

defining, due to renormalizability, the counterterm  $\delta M_\Lambda^{(2)} = \delta m_\Lambda^2 = -\delta \Sigma_\Lambda(m^2)$  and the dependence of the renormalized mass on the cutoff  $\lambda$

$$m_\lambda^2 = m^2 + \delta m_\lambda^2 = m^2 - \delta \Sigma_\lambda. \quad (60)$$

We calculate explicitly the self-energy term. The **electron** energy correction contains several terms

$$\delta p_\lambda^- = \langle p', s' | H - H_0 | p, s \rangle = \left( \sum_{n=1}^3 \delta p_{\lambda n}^- \right) \cdot \delta^{(3)}(p - p') \delta_{ss'} . \quad (61)$$

The first term comes from the commutator  $[\eta(1), H_{ee\gamma}]$

$$\delta p_{1\lambda}^- = - \int_{l_\lambda}^\infty \langle [\eta^{(1)}, H_{ee\gamma}] \rangle_{selfenergy} dl' = - \frac{\delta \Sigma_{1\lambda}(p)}{p^+}; \quad (62)$$

it reads, cf. eq. (223) in Appendix D,

$$\begin{aligned} \delta p_{1\lambda}^- &= e^2 \int \frac{d^2 k^\perp dk^+}{2(2\pi)^3} \frac{\theta(k^+)}{k^+} \theta(p^+ - k^+) \\ &\quad \times \Gamma^i(p - k, p, -k) \Gamma^i(p, p - k, k) \frac{1}{p^- - k^- - (p - k)^-} \times (-R) . \end{aligned} \quad (63)$$

This term explicitly depends on the cutoff  $\lambda$  through the  $f$ -function, that plays the role of a regulator in the loop integration

$$R_\lambda = f_{p,k,\lambda}^2 = \exp \left\{ -2 \left( \frac{\Delta_{p,k}}{\lambda} \right)^2 \right\} . \quad (64)$$

Eq. (63) corresponds to the first diagram in fig. (2).

Two instantaneous diagrams, the second and third in fig. (2), contribute cutoff independent (constant) terms. Formally one can write

$$\frac{dp^-}{dl} = \langle [\eta^{(2)}, H_0] \rangle_{selfenergy} = \langle \hat{O} \hat{O}^+ \rangle \frac{dV_{pp'}^{inst}}{dl}, \quad (65)$$

where  $\langle \hat{O} \hat{O}^+ \rangle$  stands for both the fermion and boson contraction (i.e.  $\langle b_p b_p^+ \rangle = \theta(p^+)$  and  $\langle a_k a_k^+ \rangle = \theta(k^+)$ , respectively); and  $V_{pp'}^{inst}(l) = f_{pp'}(l) V^{inst}(l=0)$  is defined in (). This gives rise to

$$\delta p_{\lambda n} = \langle \hat{O} \hat{O}^+ \rangle \cdot V_{pp}^{inst}(l) = \langle \hat{O} \hat{O}^+ \rangle \cdot V^{inst}(l=0) \quad (66)$$

for  $n = 2, 3$ . This means that  $\delta p_{\lambda n}$  defines together with  $\delta p_{\lambda 1}(l=0)$  the initial condition for the total energy correction, eq. (61).

In perturbative theory the instantaneous diagrams arise from normal-ordering Hamiltonian at  $l = 0$ , and, in principle, must accompany the first diagram for any  $l$ . In what follows we use for the instantaneous terms the same regulator  $R$ , eq. (64),

$$\begin{aligned}\delta p_{2\lambda}^- &= e^2 \int \frac{d^2 k^\perp dk^+}{2(2\pi)^3} \frac{\theta(k^+)}{k^+} \frac{\sigma^i \sigma^i}{[p^+ - k^+]} \times (-R) \\ \delta p_{3\lambda}^- &= e^2 \int \frac{d^2 k^\perp dk^+}{2(2\pi)^3} \theta(k^+) \frac{1}{2} \left( \frac{1}{[p^+ - k^+]^2} - \frac{1}{[p^+ + k^+]^2} \right) \times (-R) .\end{aligned}\quad (67)$$

We define the set of coordinates

$$\begin{aligned}x &= \frac{k^+}{p^+} \\ k &= (xp^+, xp^\perp + \kappa^\perp) ,\end{aligned}\quad (68)$$

where  $p = (p^+, p^\perp)$  is the external electron momentum. Then the electron self energy diagrams, fig. (2), cf. also eq. (227) in Appendix D, contribute

$$\begin{aligned}p^+ \delta p_{1\lambda}^- &= -\frac{e^2}{8\pi^2} \int_0^1 dx \int d\kappa_\perp^2 \\ &\quad \times \left[ \frac{p^2 - m^2}{\kappa_\perp^2 + f(x)} \left( \frac{2}{[x]} - 2 + x \right) - \frac{2m^2}{\kappa_\perp^2 + f(x)} + \left( \frac{2}{[x]^2} + \frac{1}{[1-x]} \right) \right] \times (-R) \\ f(x) &= xm^2 - x(1-x)p^2\end{aligned}\quad (69)$$

and

$$\begin{aligned}p^+ \delta p_{2\lambda}^- &= \frac{e^2}{8\pi^2} \int_0^\infty dx \int d\kappa_\perp^2 \left( \frac{1}{[x][1-x]} \right) \times (-R) \\ &\rightarrow \frac{e^2}{8\pi^2} \int_0^1 dx \int d\kappa_\perp^2 \left( \frac{1}{[x]} \right) \times (-R) \\ p^+ \delta p_{3\lambda}^- &= \frac{e^2}{8\pi^2} \int_0^\infty dx \int d\kappa_\perp^2 \left( \frac{1}{[1-x]^2} - \frac{1}{(1+x)^2} \right) \times (-R) \\ &\rightarrow \frac{e^2}{8\pi^2} \int_0^1 dx \int d\kappa_\perp^2 \left( \frac{2}{[x]^2} \right) \times (-R) ;\end{aligned}\quad (70)$$

for details we refer to Appendix D. Note, that the transformation in the integrals over  $x$  is performed *before* the regulator is taken into account [5]. (In the second integral the electron momentum is replaced by the gluon one due to momentum conservation). The brackets ' $[ ]$ ' denote the principle value prescription, defined later in eq. (76).

The loop integral over  $k$  eqs. (69) and (70) contains two types of divergencies: UV in the transversal coordinate  $\kappa^\perp$  and IR in the longitudinal component  $k^+$ . The physical value of mass must be IR-finite. We show, that the three relevant diagrams together in fact give an IR-finite value for the renormalized mass; this enables to determine *counterterms independent of longitudinal momentum*. In the wave function renormalization constant, however, the IR-singularity is still present.

Define

$$\delta_1 = \frac{p^+}{P^+} ,\quad (71)$$

where  $P = (P^+, P^\perp)$  is the positronium momentum,  $p$  the electron momentum. The transversal UV divergency is regularized through the unitary transformation done, i.e. by the regulator  $R$ , eq. (64)

$$R_\lambda = \exp \left\{ - \left( \frac{\tilde{\Delta}_{p,k}}{\lambda^2 \delta_1} \right)^2 \right\} \approx \theta(\lambda^2 \delta_1 - |\tilde{\Delta}_{p,k}|) , \quad (72)$$

where the cutoff is rescaled and defined in units of the positronium momentum  $P^+$ , namely  $\lambda \rightarrow \sqrt{2}\lambda^2/P^+$ , and  $\Delta_{p,k} = p^- - k^- - (p - k)^- = \tilde{\Delta}_{p,k}/p^+$ . The rude approximation for the exponential through a  $\theta$ -function changes the numerical coefficient within a few percent; nevertheless it is useful to estimate the integrals in eqs. (69) and (70) in this way *analytically*. From eq. (72) we have for the sum of intermediate (electron and photon) state momenta (the external electron is on-mass-shell  $p^2 = m^2$ )

$$\frac{\kappa^{\perp 2}}{[x]} + \frac{\kappa^{\perp 2} + m^2}{[1-x]} \leq \lambda^2 \delta_1 + m^2 \quad (73)$$

giving for the regulator

$$\begin{aligned} R_\lambda &= \theta(\kappa_{\lambda max}^{\perp 2} - \kappa^{\perp 2}) \theta(\kappa_{\lambda max}^{\perp 2}) \\ \kappa_{\lambda max}^{\perp 2} &= x(1-x)\lambda^2 \delta_1 - x^2 m^2 \end{aligned} \quad (74)$$

and  $\theta(\kappa_{\lambda max}^{\perp 2})$  leads to the additional condition for the longitudinal momentum

$$\begin{aligned} 0 &\leq x \leq x_{max} \\ x_{max} &= \frac{1}{1 + m^2/(\lambda^2 \delta_1)} \end{aligned} \quad (75)$$

implying that the singularity of the photon longitudinal momentum for  $x \rightarrow 1$  is regularized by the function  $R_\lambda$ . This is the case due to the nonzero fermion mass present in eq. (73) for the intermediate state with  $(1-x)$  longitudinal momentum. The IR-singularity when  $x \rightarrow 0$  is still present; it is treated by the principle value prescription [5]

$$\frac{1}{k^+} = \frac{1}{2} \left( \frac{1}{k^+ + i\varepsilon P^+} + \frac{1}{k^+ - i\varepsilon P^+} \right) , \quad (76)$$

where  $\varepsilon = 0_+$ , and  $P^+$  is the longitudinal part of the positronium momentum (used here as typical momentum in the problem being under discussion). This defines the bracket '[ ]' in eqs. (69) and (70)

$$\frac{1}{[x]} = \frac{1}{2} \left( \frac{1}{x + i\frac{\varepsilon}{\delta_1}} + \frac{1}{x - i\frac{\varepsilon}{\delta_1}} \right) . \quad (77)$$

Making use of both regularizations for transversal and longitudinal components, we have for the first diagram, eq. (69),

$$\begin{aligned} \delta m_{1\lambda}^2 &= p^+ \delta p^-|_{p^2=m^2} \\ \delta m_{1\lambda}^2 &= -\frac{e^2}{8\pi^2} \left\{ 3m^2 \ln \left( \frac{\lambda^2 \delta_1 + m^2}{m^2} \right) + \frac{\lambda^2 \delta_1}{\lambda^2 \delta_1 + m^2} \left( \frac{3}{2} \lambda^2 \delta_1 + m^2 \right) - 2\lambda^2 \delta_1 \ln \left( \frac{\lambda^2 \delta_1}{\lambda^2 \delta_1 + m^2} \frac{\delta_1}{\varepsilon} \right) \right\} \end{aligned} \quad (78)$$

Note, that the third term has the mixing UV and IR divergencies. Combining the three relevant diagrams, fig. (2), and integrating with the common regulator, one obtains for the **electron mass correction**

$$\begin{aligned}\delta m_\lambda^2 &= p^+(\delta p_1 + \delta p_2 + \delta p_3)|_{p^2=m^2} = -\delta \Sigma_\lambda(m^2) \\ \delta m_\lambda^2 &= -\frac{e^2}{8\pi^2} \left\{ 3m^2 \ln \left( \frac{\lambda^2 \delta_1 + m^2}{m^2} \right) - \frac{\lambda^2 \delta_1 m^2}{\lambda^2 \delta_1 + m^2} \right\}.\end{aligned}\quad (79)$$

The mass correction is IR-finite (that gives rise to IR-finite counterterms) and contains only a logarithmic UV-divergency. Namely, when  $\lambda \delta_1 \rightarrow \Lambda \gg m$

$$\delta m_\Lambda^2 = -\frac{3e^2}{8\pi^2} m^2 \ln \frac{\Lambda^2}{m^2}.\quad (80)$$

It is remarkable that we reproduce with the cutoff condition of eq. (73) the standard result of covariant perturbative theory calculations including its global factor 3/8. As was mentioned above, the difference in sign, as compared with the 1-loop renormalization group result, comes from scaling down from high to low energies in the method of flow equations.

The similar regularization for the intermediate state momenta in the self-energy integrals, called global cutoff scheme, was introduced by W. M. Zhang and A. Harindranath [5]. In our approach the UV-regularization, that defines the concrete form of the regulator  $R$ , arises naturally from the method of flow equations, namely from the unitary transformation performed, where the generator of the transformation is chosen as the commutator  $\eta = [H_0, H]$ . Note also, that the regulator  $R$ , eq. (72), in general is independent of the electron momentum  $p^+$  (rescaled cutoff  $\lambda \delta_1 \rightarrow \lambda$ ), and therefore is boost invariant.

For the wave function renormalization constant, eq. (58), one has

$$\left. \frac{\partial \delta p^-}{\partial p^-} \right|_{p^2=m^2} = -\frac{e^2}{8\pi^2} \int_0^1 \int d\kappa_\perp^2 \left[ \frac{2\frac{1}{x} - 2 + x}{\kappa_\perp^2 + f(x)} - \frac{x(1-x)2m^2}{(\kappa_\perp^2 + f(x))^2} \right]_{p^2=m^2} \times (-R),\quad (81)$$

that together with the regulator  $R$ , eq. (72), results

$$\begin{aligned}Z_2 &= 1 - \frac{e^2}{8\pi^2} \left\{ \ln \frac{\lambda^2 \delta_1}{m^2} \cdot \left( \frac{3}{2} - 2 \ln \frac{\delta_1}{\varepsilon} \right) + \ln \frac{\delta_1}{\varepsilon} \cdot \left( 2 - \ln \frac{\delta_1}{\varepsilon} \right) + F \left( \ln \frac{\lambda^2 \delta_1}{\lambda^2 \delta_1 + m^2}; \frac{\lambda^2 \delta_1}{\lambda^2 \delta_1 + m^2} \right) \right\} \\ F &= \ln \frac{\lambda^2 \delta_1}{\lambda^2 \delta_1 + m^2} \left( \frac{1}{2} - \ln \frac{\lambda^2 \delta_1}{\lambda^2 \delta_1 + m^2} \right) + \frac{1}{2} \frac{\lambda^2 \delta_1}{\lambda^2 \delta_1 + m^2} - 2 + 2 \int_0^{x_{max}} dx \frac{\ln x}{x-1}.\end{aligned}\quad (82)$$

As  $\lambda \delta_1 \rightarrow \Lambda \gg m$  the function  $F$  tends to a constant

$$F|_{\Lambda \gg m} = C = -\frac{3}{2} + \frac{\pi^2}{3}.\quad (83)$$

Therefore, by dropping the finite part, we obtain

$$Z_2 = 1 - \frac{e^2}{8\pi^2} \left\{ \ln \frac{\Lambda^2}{m^2} \cdot \left( \frac{3}{2} - 2 \ln \frac{1}{\varepsilon} \right) + \ln \frac{1}{\varepsilon} \left( 2 - \ln \frac{1}{\varepsilon} \right) \right\},\quad (84)$$

where we have rescaled  $\frac{\varepsilon}{\delta_1} \rightarrow \varepsilon$ . The electron wave function renormalization constant contains logarithmic UV and IR divergencies *mixed*, together with *pure* logarithmic IR divergencies. We mention, that the value of  $Z_2$  is not sensitive to what kind of regulator is applied; the same result for  $Z_2$  was obtained with another choice of regulator [5].

We proceed with renormalization to the second order in the photon sector. The diagrams that contribute to the photon self energy are shown in fig. (3). The commutator  $[\eta^{(1)}, H_{ee\gamma}]$ , corresponding to the first diagram, gives rise to (cf. eq. (235) in Appendix D)

$$\begin{aligned} \delta q_{1\lambda}^- \delta^{ij} &= \frac{1}{[q^+]} e^2 \int \frac{d^2 k^\perp dk^+}{2(2\pi)^3} \theta(k^+) \theta(q^+ - k^+) \\ &\quad \times Tr \left[ \Gamma^i(k, k - q, q) \Gamma^j(k - q, k, -q) \right] \frac{1}{q^- - k^- - (q - k)^-} \times (-R) , \end{aligned} \quad (85)$$

where momenta are given in fig. (3) , and the regulator is

$$R_\lambda = f_{q,k,\lambda}^2 = \exp \left\{ -2 \left( \frac{\Delta_{q,k}}{\lambda} \right)^2 \right\} . \quad (86)$$

In full analogy with the electron self energy this also defines the regulator for the second diagram with the instantaneous interaction, see fig. (3),

$$\delta q_{2\lambda}^- \delta^{ij} = \frac{1}{[q^+]} e^2 \int \frac{d^2 k^\perp dk^+}{2(2\pi)^3} \theta(k^+) Tr(\sigma^i \sigma^j) \left( \frac{1}{[q^+ - k^+]} - \frac{1}{[q^+ + k^+]} \right) \times (-R) . \quad (87)$$

We define the set of coordinates

$$\begin{aligned} \frac{(q - k)^+}{q^+} &= x \\ k &= ((1 - x)q^+, (1 - x)q^\perp + \kappa^\perp) \\ (q - k) &= (xq^+, xq^\perp - \kappa^\perp) , \end{aligned} \quad (88)$$

where  $q = (q^+, q^\perp)$  is the external photon momentum. Then two diagrams contribute (for details see Appendix D, eq. (239)):

$$\begin{aligned} q^+ \delta q_1^- &= -\frac{e^2}{8\pi^2} \int_0^1 dx \int d\kappa_\perp^2 \\ &\quad \times \left\{ \frac{q^2}{\kappa_\perp^2 + f(x)} (2x^2 - 2x + 1) + \frac{2m^2}{\kappa_\perp^2 + f(x)} + \left( -2 + \frac{1}{[x][1 - x]} \right) \right\} \times (-R) \\ f(x) &= m^2 - x(1 - x)q^2 \\ q^+ \delta q_2^- &= \frac{e^2}{8\pi^2} \int_0^\infty dx \int d\kappa_\perp^2 \left( \frac{1}{[1 - x]} - \frac{1}{1 + x} \right) \times (-R) \\ &\rightarrow -\frac{e^2}{8\pi^2} \int_0^1 dx \int d\kappa_\perp^2 \frac{2}{[x]} \times (-R) . \end{aligned} \quad (89)$$

Note, that the transformation in the second integral is done *before* the regularization (by regulator the  $R$ ) is performed [5].

Making use of the same approximation for the regulator as in the electron sector, we obtain for the sum of intermediate (two electron) state momenta

$$\begin{aligned} \frac{\kappa_\perp^2 + m^2}{x} + \frac{\kappa_\perp^2 + m^2}{1 - x} &\leq \lambda^2 \delta_2 \\ \delta_2 &= \frac{q^+}{P^+} , \end{aligned} \quad (90)$$

where the photon is put on mass-shell  $q^2 = 0$  and the rescaled cutoff  $\lambda \rightarrow \sqrt{2}\lambda^2/P^+$  has been used. This condition means for the transversal integration

$$\begin{aligned} R_\lambda &= \theta(\kappa_{\lambda max}^{\perp 2} - \kappa^{\perp 2}) \theta(\kappa_{\lambda max}^{\perp 2}) \\ \kappa_{\lambda max}^{\perp 2} &= x(1-x)\lambda^2\delta_2 - m^2 \end{aligned} \quad (91)$$

and for the longitudinal integration

$$\begin{aligned} x_1 &\leq x \leq x_2 \\ x_1 &= \frac{1-r}{2} \approx \frac{m^2}{\lambda^2\delta_2} \\ x_2 &= \frac{1+r}{2} \approx 1 - \frac{m^2}{\lambda^2\delta_2} \\ r &= \sqrt{1 - \frac{4m^2}{\lambda^2\delta_2}}, \end{aligned} \quad (92)$$

where the approximate value is when  $m \ll \lambda$ . This shows that the condition of eq. (90) for two electrons with masses  $m$  removes the light-front infrared singularities from  $x \rightarrow 0$  and  $x \rightarrow 1$ . Thus, both UV and IR divergencies are regularized by the regulator  $R$ , eq. (91).

The mass correction arising from the first diagram, eq. (89), is

$$\delta m_{1\lambda}^2 = \frac{e^2}{8\pi^2} \frac{2}{3} \lambda^2 \delta_2 \left(1 - \frac{4m^2}{\lambda^2\delta_2}\right)^{3/2}. \quad (93)$$

Combining together both diagrams with the same regulator, eq. (89), we obtain

$$\delta m_\lambda^2 = \frac{e^2}{8\pi^2} \left( \frac{5}{3} \lambda^2 \delta_2 r - \frac{8}{3} m^2 r - 2m^2 \ln \frac{1+r}{1-r} \right), \quad (94)$$

where  $r$  is defined in eq. (92). The result shows that the mass correction involves the quadratic and logarithmic UV divergencies, i.e. as  $\lambda\delta_2 \rightarrow \Lambda \gg m$

$$\delta m_\Lambda^2 = \frac{e^2}{8\pi^2} \left( \frac{5}{3} \Lambda^2 - 2m^2 \ln \frac{\Lambda^2}{m^2} \right). \quad (95)$$

The wave function renormalization constant is defined through

$$\left. \frac{\partial \delta q^-}{\partial q^-} \right|_{q^2=0} = -\frac{e^2}{8\pi^2} \int_0^1 dx \int d\kappa_\perp^2 \left\{ \frac{2x^2 - 2x + 1}{\kappa_\perp^2 + f(x)} + \frac{2m^2 x(1-x)}{(\kappa_\perp^2 + f(x))^2} \right\} \Big|_{q^2=0} \times (-R), \quad (96)$$

that, with the regulator  $R$ , eq. (91), results

$$Z_2 = 1 - \frac{e^2}{8\pi^2} \left( -\frac{2}{3} \ln \frac{1+r}{1-r} + \frac{10}{9} r + \frac{8}{9} \frac{m^2}{\lambda^2\delta_2} r \right). \quad (97)$$

The photon wave function renormalization constant contains only logarithmic UV divergency, indeed as  $\lambda\delta_2 \rightarrow \Lambda \gg m$

$$Z_2 = 1 - \frac{e^2}{8\pi^2} \left( -\frac{2}{3} \ln \frac{\Lambda^2}{m^2} \right) \quad (98)$$

and is free of IR divergencies (as is expected from the form of the regulator  $R$ , eq. (90)).



### 3.3 Renormalized theory to the order $e^2$ (at the scale $\lambda$ ) and Feynmann rules

We have completed the renormalization of light front QED (LFQED) to the second order, what enables us to formulate the diagrammatic rules for the perturbative expansions in  $e_0$ . We formulate the Feynman rules (in light front frame) for the renormalized  $x^+$ -ordered LF Hamiltonian. All matrix elements of the renormalized Hamiltonian  $H_r$ , namely the interacting part of  $H_r$  with the corresponding LFQED vertices, are listed in fig. (1) together with their diagrams. The diagrammatic rules are obtained by direct calculation of matrix elements between free particle states.

The two last diagrams in fig. (1) correspond to the generated interactions, arising from the renormalized Hamiltonian considered in different sectors. Note, that the existence of the  $|ee\gamma\rangle$ -vertex in the renormalized Hamiltonian at any finite cutoff  $\lambda$  prevents the generated interaction to have small energy denominators  $\frac{1}{\Delta_1}, \frac{1}{\Delta_2}$  (the  $|ee\gamma\rangle$ -vertex appears through the exponential factor in the generated interaction). We mention, that the problem of small energy denominators in generated terms also is solved in the similarity scheme of Glazek and Wilson [2].

The two-component LF theory, introduced by Zhang and Harindranath [5], as compared with the four-component formalism of Lepage and Brodsky is formulated purely in terms of physical degrees of freedom, so that each term corresponds to a real dynamical process. This means, for instance, that the first of the two generated term diagrams must be taken into account in low energy  $ee$ -scattering, and the second one describes Compton scattering at low energies. Further, we use the instantaneous and generated interactions, namely the second and the forth diagrams in  $e\bar{e}$  sector, together with the perturbative theory contribution to the order  $O(e^2)$  of  $ee\gamma$  vertices (in the same sector), to calculate the mass of the positronium bound state. The rules to write the expression of perturbative expansions from diagrams are given in [5].

Note, that the similarity function  $f_{p_i p_f \lambda}$  in each term restricts the energy differences between initial and final states to be below the UV cutoff  $\lambda$ , and thus restricts the renormalized Hamiltonian to act in the low energy sector. Therefore, for the UV cutoff  $\lambda \ll \Lambda$ , we associate the renormalized Hamiltonian with the effective Hamiltonian describing the physics of low energy.

Another aspect of the flow equation method is connected with renormalization group. Namely, the function  $f_{p_i p_f \lambda}$  in the  $ee\gamma$ -vertex plays the role of UV (and partially IR) regulator in the self energy integrals (see before), so that the regularization prescription of divergent integrals originates from the method of flow equations itself. Moreover, as we will further show, the energy correction (i.e. mass correction and wave function renormalization constant), obtained from the flow equation method, coincide up to the overall sign with the 1-loop renormalization group result. This is the remarkable result, indicating the equivalence of flow equations and Wilson's renormalization.

At last we mention that in the diagrammatic rules given in fig. (1) we explicitly write the dependence of the electron (photon) mass on the cutoff, namely

$$m_\lambda^2 = m_0^2 - \delta\Sigma_\lambda, \quad (99)$$

where  $\delta\Sigma_\lambda$  is the self energy term to the order  $O(e^2)$ , ( $m_0 = 0$  for a photon). Whereas we drop the subscript  $\lambda$  for the polarization vectors  $\varepsilon$  and spinors  $\chi$ . To the next order  $O(e^3)$  one has  $e_\lambda = e_0^2(1 + O(e_\lambda^2))$ .

In the next section we use these diagrammatic rules of LFQED renormalized Hamiltonian, fig. (1), to calculate perturbatively electron-positron interaction, namely the contribution from

perturbative photon exchange and electron (photon) self energy correction, which latter enables to find the corresponding physical masses.

## 4 Positronium's fine structure

### 4.1 LF Perturbative theory

The scattering  $|e\bar{e}\rangle$  states are also needed in bound state calculations. Using the propagator techniques we include these states where required. We exploit the perturbative theory in the coupling constant  $e$ , using the Feynman rules of the renormalized theory fig. (1).

The first order renormalized  $ee\gamma$ -vertex  $f_{p_i p_f} H_{can}^{ee\gamma}$  contributes to the second order to the  $|e\bar{e}\rangle$  interaction term and to the electron (photon) mass renormalization. Physically, it is the perturbative photon exchange (photon emission and absorbtion in the case of electron mass renormalization), with the energy widths of the photon restricted by the function  $f_{p_i p_f, \lambda}$ .

#### 4.1.1 The electron-positron interaction

According to the light-front Feynman rules the perturbative photon exchange gives rise to the following second order  $|e\bar{e}\rangle$  interaction in the exchange channel

$$V(l) = g_1(l)g_2(l)M_{2ii} \cdot \left\{ \frac{\theta(q^+)}{q^+} \frac{1}{p_i^- - p_k^-} + \frac{\theta(-q^+)}{(-q^+)} \frac{1}{p_i^- - p_k^-} \right\}, \quad (100)$$

where  $g_i$  stands schematically for the coupling constants in both vertices, namely  $g_{p_1 p_2 \lambda} = e f_{p_1 p_2 \lambda}$  and  $M_{2ii}$  defines the spin structure of the interaction, coming from the corresponding structure of the  $ee\gamma$ -vertex; and the two terms in the curly brackets represent two different  $x^+$  (time) orderings of the photon exchange with the momentum  $q$ , giving rise to the two different intermediate states with momenta  $p_k, p_i$  corresponds to the initial state. Explicitly one has

In the **exchange channel**

$$\begin{aligned} V_{PT}^{(ex)}(l) &= -e^2 (f_{-p_4, -p_2}(l) \chi_{\bar{s}_2}^+ \Gamma^i(-p_4, -p_2, -q) \chi_{\bar{s}_4}) (f_{p_1, p_3}(l) \chi_{s_3}^+ \Gamma^i(p_1, p_3, q) \chi_{s_1}) \\ &\times \left[ \frac{\theta(p_1^+ - p_3^+)}{(p_1^+ - p_3^+)} \frac{1}{p_i^- - p_3^- - p_2^- - q^-} + \frac{\theta(p_3^+ - p_1^+)}{(p_3^+ - p_1^+)} \frac{1}{p_i^- - p_1^- - p_4^- + q^-} \right] \bar{\delta}_{q, p_1 - p_3} \end{aligned} \quad (101)$$

with the initial state momentum  $p_i = P = (P^+, P^\perp)$  and momentum transfer  $q = p_1 - p_3$ , and

$$P^- = \frac{P^{\perp 2} + M_N^2}{P^+}, \quad (102)$$

where  $M_N$  is the mass of positronium bound state. In the light-front frame

$$\begin{aligned} -(P^- - p_3^- - p_2^- - (p_1 - p_3)^-) \theta(p_1^+ - p_3^+) &= (P^- - p_1^- - p_4^- + (p_1 - p_3)^-) \theta(p_3^+ - p_1^+) \\ &= \frac{\tilde{\Delta}_3}{P^+(x - x')} \end{aligned} \quad (103)$$

holds, giving for the rescaled potential  $V \rightarrow P^{+2}V$  rise to

$$V_{PT, \lambda}^{(ex)} = -e^2 N_1 \frac{1}{\tilde{\Delta}_3} \exp \left( -\frac{(\Delta_1^2 + \Delta_2^2)}{\lambda^4} \right), \quad (104)$$

where  $N_1$  is defined in eq. (53) and

$$\tilde{\Delta}_3 = (k_\perp - k'_\perp)^2 + \frac{1}{2}(x - x')A + |x - x'| \left( \frac{1}{2}(M_0^2 + M_0'^2) - M_N^2 \right) \quad (105)$$

$$A = (k_\perp^2 + m^2) \left( \frac{1}{1-x} - \frac{1}{x} \right) + (k'_\perp^2 + m^2) \left( \frac{1}{x'} - \frac{1}{1-x'} \right) ,$$

Here the cutoff  $\lambda$  is defined in units of  $P^+$ .

Because of the absence of Z-graphen in light-front formalism (corresponding to negative  $p^+$ ), only one term contribute to the **annihilation channel**, namely

$$\begin{aligned} V_{PT}^{(an)}(l) &= e^2 (f_{p_1, -p_2}(l) \chi_{\bar{s}_2}^+ \Gamma^i(p_1, -p_2, q) \chi_{s_1}) (f_{-p_4, p_3}(l) \chi_{s_3}^+ \Gamma^i(-p_4, p_3, -q) \chi_{\bar{s}_4}) \\ &\times \left[ \frac{1}{(p_1^+ + p_2^+)} \frac{1}{p_i^- - q^-} \right] \bar{\delta}_{q, p_1 + p_2} , \end{aligned} \quad (106)$$

where  $p_i^- = P^-$  and the momentum transfer is  $q = p_1 + p_2$ . This gives rise for the rescaled potential  $V \rightarrow P^{+2}V$  in the light-front frame, to the expression

$$V_{PT, \lambda}^{(an)} = e^2 N_2 \frac{1}{M_N^2} \exp \left\{ -\frac{(M_0^4 + M_0'^4)}{\lambda^4} \right\} , \quad (107)$$

where  $N_2$  and the variables  $\Delta_1, \Delta_2$  and  $M_0^2, M_0'^2$  are defined in eq. (53).

#### 4.1.2 Mass renormalization

Following light-cone rules the perturbative energy correction of the electron with momentum  $p$ , coming from the emission and absorption of a photon with momentum  $k$ , is

$$\begin{aligned} \delta \tilde{p}_{1\lambda}^- &= \int \frac{d^2 k^\perp dk^+}{2(2\pi)^3} \frac{\theta(k^+)}{k^+} \theta(p^+ - k^+) g_{p-k, p, \lambda} \Gamma_\lambda^i(p - k, p, -k) g_{p, p-k, \lambda} \Gamma_\lambda^i(p, p - k, k) \\ &\times \frac{1}{p^- - k^- - (p - k)^-} , \end{aligned} \quad (108)$$

where  $g_{ee\gamma}$ -coupling constant restricts the energy of the photon. Making use of the explicit form for the coupling, one has

$$\begin{aligned} \delta \tilde{p}_{1\lambda}^- &= e^2 \int \frac{d^2 k^\perp dk^+}{2(2\pi)^3} \frac{\theta(k^+)}{k^+} \theta(p^+ - k^+) \\ &\times \Gamma_\lambda^i(p - k, p, -k) \Gamma_\lambda^i(p, p - k, k) \frac{1}{p^- - k^- - (p - k)^-} \times (R) , \end{aligned} \quad (109)$$

where  $R = f_{pk\lambda}^2$  plays the role of regulator. This expression coincide up to the overall sign with the energy correction obtained in the previous section from the flow equations method.

Two instantaneous diagramms, arising from the normal-ordering Hamiltonian, must be added to the first term with the same regulator  $R$ . Then the full perturbative energy correction  $\delta \tilde{p}_\lambda^- = \delta \tilde{p}_{1\lambda}^- + \delta \tilde{p}_{2\lambda}^- + \delta \tilde{p}_{3\lambda}^-$  is

$$\delta \tilde{p}_\lambda^- = -\delta p_\lambda^- \quad (110)$$

where  $\delta p_\lambda^-$  is defined in eq. (61). This means for the perturbative mass correction

$$\delta m_\lambda^{PT2} = \delta \Sigma_\lambda \quad (111)$$

and the self-energy term  $\delta \Sigma_\lambda$  is given in eq. (79).

We combine the renormalized to the second order mass, eq. (60), and the perturbative correction, eq. (111), to obtain the total physical mass to the order  $O(e^2)$

$$m_e^2 = m_\lambda^2 + \delta m^2 = (m^2 + \delta \Sigma_\lambda) - \delta \Sigma_\lambda = m^2 + O(e^4). \quad (112)$$

This means, that to the second order  $O(e^2)$  the physical electron mass is, up to a finite part, equal to the bare electron mass, that stands in the free (canonical) Hamiltonian.

Along the same line one can do for the photon mass.

## 4.2 Bound state perturbative theory

In this subsection we define bound state perturbative theory (BSPT).

First introduce instead of the front parametrization, used before for the single-particle momenta, fig. (4), the instant form

$$\begin{aligned} p_{1\mu} &= (xP^+, xP^\perp + k^\perp, p_1^-) \xrightarrow{J(p)} p_{1\mu} = (k_z, k^\perp, p_1^0) = (\vec{p}_1, E_1) \\ p_{2\mu} &= ((1-x)P^+, (1-x)P^\perp - k^\perp, p_2^-) \longrightarrow p_{2\mu} = (-k_z, -k^\perp, p_2^0) = (\vec{p}_2, E_2) \\ E_i &= \sqrt{\vec{p}^2 + m^2}, \quad i = 1, 2 \\ x &= \frac{E_1 + k_z}{E_1 + E_2} = \frac{1}{2} \left( 1 + \frac{k_z}{\sqrt{\vec{p}^2 + m^2}} \right) \end{aligned} \quad (113)$$

and for the momenta  $p_3, p_4$  the same, but with prime over  $x, k_z, k^\perp$ ; here  $x$  is the light-front fraction of the electron momentum, and  $J(p)$  is the Jacobian of the transformation:

$$J(p) = \frac{dx}{dk_z} = \frac{k_\perp^2 + m^2}{2(\vec{p}^2 + m^2)^{3/2}}. \quad (114)$$

The new coordinate system corresponds to the center of mass (c.m.) frame, i.e.  $\vec{P} = 0$ . Note, that in both systems electron and positron are moving along paths on the light-cone (against each other in the instant case  $P_3 = k_{z1} + k_{z2} = 0$ ), while the connection between the two frames is obtained by boost.

The light-front bound state equation (see later) is boost and frame invariant; therefore it can be solved in the c.m. frame, producing the same spectrum as in the front form. We shall see that the electron-positron potential, arising from the renormalized to the second order Hamiltonian, is drastically simplified in the c.m. instant frame. Also, this system is usefull in practical aspect: in making obvious the rotational symmetry, restored in the nonrelativistic limit and manifest in the spectrum.

We stress an important consequence of this change of coordinates. Since the total light-front momentum  $P^+ = P_0 + P_3$  is conserved (i.e. both ‘in’  $|x, k^\perp\rangle$  and ‘out’  $|x', k'^\perp\rangle$  Fock states have the same  $P^+$ ) and  $P_3 = 0$  in instant form, we have *conservation of the total energy*  $P_0 = P'_0$ , or, in other words,  $E_1 + E_2 = E_3 + E_4$ . For the 3-d momentum this means

$$\vec{p}^2 = \vec{p}'^2. \quad (115)$$

This condition we will further exploit in what follows.

Define now **BSPT**. We choose the leading order electron-positron potential in such a form to simplify positronium bound state calculations. This means, that this potential contributes the leading order term to the positronium mass, and perturbative theory with respect to the difference between the total second order  $|e\bar{e} >$  interaction, calculated before with the renormalized Hamiltonian  $H_r$  to  $O(e^2)$ , and the leading order potential converges. This scheme we call BSPT. Surely, our choice is motivated by the form of the renormalized to the second order interaction to insure convergence of BSPT.

We define the second order renormalized electron-positron potential  $\langle e(3)\bar{e}(4) | \hat{V}_{coul} | e(1)\bar{e}(2) \rangle$  to the leading order of BSPT in the form of pure perturbative one photon exchange, explicitly as the Coulomb interaction

$$V_{coul} = -\frac{16e^2m^2}{(k_\perp - k'_\perp)^2 + (k_z - k'_z)^2} = -\frac{16e^2m^2}{(\vec{p} - \vec{p}')^2} . \quad (116)$$

This means that the corresponding leading order Hamilton operator in the  $|e\bar{e} >$  sector is

$$H^{(0)} = h + \hat{V}_{coul} , \quad (117)$$

where  $h$  is the free part, defined in eq. (28). The wave functions are given as the solution of Schrödinger equation

$$H^{(0)}|\psi_N(P) > = E_N|\psi_N(P) > , \quad (118)$$

where  $P$  is the positronium momentum, and the eigenvalues and eigenfunctions for the positronium bound state are defined in standard way of light front frame

$$\begin{aligned} E_N &= \frac{P_\perp^2 + M_N^2}{P^+} \\ |\psi_N(P) > &= \sum_{s_1 s_2} \int_{p_1 p_2} \sqrt{p_1^+ p_2^+} 2(2\pi)^3 \delta^{(3)}(P - p_1 p_2) \tilde{\Phi}_N(x k_\perp s_1 s_2) b_{s_1}^+(p_1) d_{s_2}^+(p_2) |0 > \\ &\sum_{s_1 s_2} \frac{\int d^2 k_\perp \int_0^1 dx}{2(2\pi)^3} \tilde{\Phi}_N^*(x k_\perp s_1 s_2) \tilde{\Phi}'_N(x k_\perp s_1 s_2) = \delta_{NN'} \end{aligned} \quad (119)$$

$M_N$  stands for the leading order mass of positronium. Combining the definitions for the wave function and the energy with the Schrödinger equation, we obtain

$$\left[ M_N^2 - \frac{k_\perp'^2 + m^2}{x'(1-x')} \right] \tilde{\Phi}_N(x' k'_\perp s_3 s_4) = \sum_{s_1 s_2} \frac{\int d^2 k_\perp \int_0^1 dx}{2(2\pi)^3} V_{coul} \tilde{\Phi}_N(x k_\perp s_1 s_2) , \quad (120)$$

or, after change of coordinates according to eq. (113),

$$\left( M_N^2 - 4(\vec{p}^2 + m^2) \right) \Phi_N(\vec{p} s_3 s_4) = \sum_{s_1 s_2} \frac{\int d^3 p \sqrt{J(p)J(p')}}{2(2\pi)^3} V_{coul}(\vec{p}, \vec{p}') \Phi_N(\vec{p} s_1 s_2) , \quad (121)$$

where the wave function was redefined to have the norm

$$\sum_{s_1 s_2} \int d^3 p \Phi_N^*(\vec{p} s_1 s_2) \Phi'_N(\vec{p} s_1 s_2) = \delta_{NN'} . \quad (122)$$

We aim to obtain the nonrelativistic Schrödinger equation for positronium. Note, that in the nonrelativistic limit  $\frac{\vec{p}^2}{m^2} \ll 1$  we have

$$\begin{aligned} \sqrt{J(p)J(p')} &\approx \frac{1}{2m} \left( 1 - \frac{\vec{p}^2 + (k_z^2 + k_z'^2)}{2m^2} \right) \\ M_N &= (2m + B_N)^2 \approx 4m^2 + 4mB_N^{(0)} , \end{aligned} \quad (123)$$

where we have introduced the leading order binding energy  $B_N^{(0)}$ . Then to the leading order the bound state equation for positronium is

$$\left(\frac{\vec{p}^2}{m} - B_N\right) \Phi_N(\vec{p}' s_3 s_4) = -\sum_{s_1 s_2} \int d^3 p \left(\frac{1}{2m} \frac{1}{2(2\pi)^3} \frac{1}{4m} V_{coul}\right) \Phi_N(\vec{p} s_1 s_2) . \quad (124)$$

Making use of the explicit form for the Coulomb potential, eq. (116), we obtain the equation that determines the leading order bound state wave function:

$$\left(\frac{\vec{p}'^2}{m} - B_N\right) \Phi_\mu(\vec{p}') = \frac{\alpha}{2\pi^2} \int \frac{d^3 p}{(\vec{p} - \vec{p}')^2} \Phi_\mu(\vec{p}) \quad (125)$$

with

$$\Phi_N = \Phi_{\mu, s_e, s_{\bar{e}}}(\vec{p}' s_3 s_4) = \Phi_\mu(\vec{p}') \delta_{s_e s_3} \delta_{s_{\bar{e}} s_4} . \quad (126)$$

This is the standard nonrelativistic Schrödinger equation for positronium. Its solution is characterized by  $\mu=(n, l, m)$ , the usual principal and angular momentum quantum numbers. The wave functions are given through the hyperspherical harmonics

$$\begin{aligned} Y_\mu(\Omega) &= \frac{(e_n^2 + \vec{p}^2)^2}{4 e_n^{5/2}} \Phi_\mu \\ Y_\mu &= Y_{n,l,m} = f_{n,l}(\omega) Y_{l,m}(\theta, \phi) \\ B_N &= -\frac{m\alpha^2}{4n^2}, \quad e_n = \frac{m\alpha}{2n} \end{aligned} \quad (127)$$

and for the binding energy one has the standard nonrelativistic expression for positronium bound state to  $O(e^2)$ . For sake of completeness we write the coordinates used in the solution

$$\begin{aligned} (e_n^2 = -mB_N, \vec{p}) &\longrightarrow (u_0, \vec{u}) \\ u_0 &= \cos \omega = \frac{e_n^2 - \vec{p}^2}{e_n^2 + \vec{p}^2} \\ \vec{u} &= \frac{\vec{p}}{|\vec{p}|} \sin \omega = \frac{2e_n \vec{p}}{e_n^2 + \vec{p}^2} , \end{aligned} \quad (128)$$

but, for details, refer to [6].

The electron-positron interaction arising from the renormalized to the  $O(e^2)$  Hamiltonian is given as a sum of two contributions from exchange and annihilation channels  $V = V_{exch} + V_{ann}$  (see explicitly later). We introduce the potential, arising in the nonrelativistic Schrödinger equation, eq. (125),

$$\tilde{V}(\vec{p}' s_3 s_4; \vec{p} s_1 s_2) = \lim_{\frac{\vec{p}^2}{m^2} < 1} \frac{\sqrt{J(p)J(p')}}{2(2\pi)^3} \frac{1}{4m} (V_{exch} + V_{ann}) . \quad (129)$$

Then we define BSPT with respect to the difference

$$\delta V = \tilde{V}(\vec{p}' s_3 s_4; \vec{p} s_1 s_2) - \left(-\frac{\alpha}{2\pi^2}\right) \frac{1}{(\vec{p} - \vec{p}')^2} \delta_{s_1 s_3} \delta_{s_2 s_4} , \quad (130)$$

where the leading order contribution is defined in eq. (127). Note, that, to define the Coulomb potential, we have taken only the first term of the Jacobian's  $J(p)$  nonrelativistic expansion, i.e. the  $e\bar{e}$  interaction to the leading order of BSPT.

In what follows we use the matrix elements of  $\delta V$ , defined as

$$\langle \Phi_{nlm} | \delta V | \Phi_{nlm} \rangle = \int d^3p d^3p' \Phi_{nlm}^*(\vec{p}) \delta V \Phi_{nlm}(\vec{p}') , \quad (131)$$

where  $\Phi_{nlm}$  are the Coulomb wave functions given above.

### 4.3 The renormalized electron-positron interaction in light-front and instant form frames

We summarize all together the second order  $|e\bar{e}\rangle$  interaction in the exchange and annihilation channels, i.e., written in the **front form frame**:

$$\begin{aligned} V_{exch} &= V_{\lambda}^{exch} + V^{PT} = V_{\lambda}^{gen} + V_{\lambda}^{inst} + V_{\lambda}^{PT} \\ V_{ann} &= V_{\lambda}^{ann} + V^{PT} = V_{\lambda}'^{gen} + V_{\lambda}'^{inst} + V_{\lambda}'^{PT} , \end{aligned} \quad (132)$$

where the generated, instantaneous and perturbative theory interactions (rescaled, i.e.  $V \rightarrow P^{+2}V$ ) are given correspondingly in the **exchange channel**

$$\begin{aligned} V_{\lambda}^{gen} &= -e^2 N_{1,\lambda} \cdot \frac{1}{2} \left( \frac{1}{\tilde{\Delta}_1} + \frac{1}{\tilde{\Delta}_2} \right) \left( 1 - e^{-2 \frac{\tilde{\Delta}_1}{\lambda^2} \cdot \frac{\tilde{\Delta}_2}{\lambda^2}} \right) e^{-\left( \frac{M_0^2 - M_0'^2}{\lambda^2} \right)^2} \\ V_{\lambda}^{inst} &= -\frac{4e^2}{(x-x')^2} \delta_{s_1 s_3} \delta_{s_2 s_4} \cdot e^{-\left( \frac{M_0^2 - M_0'^2}{\lambda^2} \right)^2} \\ V_{\lambda}^{PT} &= -e^2 N_{1,\lambda} \cdot \frac{1}{\tilde{\Delta}_3} e^{-\left( \left( \frac{\tilde{\Delta}_1}{\lambda^2} \right)^2 + \left( \frac{\tilde{\Delta}_2}{\lambda^2} \right)^2 \right)} \end{aligned} \quad (133)$$

in the **annihilation channel**

$$\begin{aligned} V_{\lambda}^{gen} &= e^2 N_{2,\lambda} \cdot \frac{1}{2} \left( \frac{1}{M_0^2} + \frac{1}{M_0'^2} \right) \left( 1 - e^{-2 \frac{M_0^2}{\lambda^2} \cdot \frac{M_0'^2}{\lambda^2}} \right) e^{-\left( \frac{M_0^2 - M_0'^2}{\lambda^2} \right)^2} \\ V_{\lambda}^{inst} &= 4e^2 \delta_{s_1 \bar{s}_3} \delta_{s_2 \bar{s}_4} \cdot e^{-\left( \frac{M_0^2 - M_0'^2}{\lambda^2} \right)^2} \\ V_{\lambda}^{PT} &= e^2 N_{2,\lambda} \cdot \frac{1}{M_N^2} e^{-\left( \left( \frac{M_0^2}{\lambda^2} \right)^2 + \left( \frac{M_0'^2}{\lambda^2} \right)^2 \right)} , \end{aligned} \quad (134)$$

where in the light-front frame, see eq. (194) in Appendix B,

$$\begin{aligned} N_{1,\lambda} &= \delta_{s_1 s_3} \delta_{s_2 s_4} T_1^{\perp} \cdot T_2^{\perp} - \delta_{s_1 \bar{s}_2} \delta_{s_1 \bar{s}_3} \delta_{s_2 \bar{s}_4} 2m^2 \frac{(x-x')^2}{xx'(1-x)(1-x')} \\ &\quad + im\sqrt{2}(x'-x) \left[ \delta_{s_1 \bar{s}_3} \delta_{s_2 s_4} \frac{s_1}{xx'} T_1^{\perp} \cdot \varepsilon_{s_1}^{\perp} + \delta_{s_1 s_3} \delta_{s_2 \bar{s}_4} \frac{s_2}{(1-x)(1-x')} T_2^{\perp} \cdot \varepsilon_{s_2}^{\perp} \right] \\ N_{2,\lambda} &= \delta_{s_1 \bar{s}_2} \delta_{s_3 \bar{s}_4} T_3^{\perp} \cdot T_4^{\perp} + \delta_{s_1 s_2} \delta_{s_3 s_4} \delta_{s_1 s_3} 2m^2 \frac{1}{xx'(1-x)(1-x')} \\ &\quad + im\sqrt{2} \left[ \delta_{s_3 \bar{s}_4} \delta_{s_1 s_2} \frac{s_1}{x(1-x)} T_3^{\perp} \cdot \varepsilon_{s_1}^{\perp} - \delta_{s_3 s_4} \delta_{s_1 \bar{s}_2} \frac{s_3}{x'(1-x')} T_4^{\perp} \cdot \varepsilon_{s_4}^{\perp*} \right] \\ \varepsilon_s^i &= -\frac{1}{\sqrt{2}}(s, i) \end{aligned} \quad (135)$$

and

$$\begin{aligned}
T_1^i &= - \left[ 2 \frac{(k_\perp - k'_\perp)^i}{(x - x')} + \frac{k_\perp^i(s_2)}{(1 - x)} + \frac{k_\perp'^i(\bar{s}_2)}{(1 - x')} \right] ; & T_2^i &= 2 \frac{(k_\perp - k'_\perp)^i}{(x - x')} - \frac{k_\perp^i(s_1)}{x} - \frac{k_\perp'^i(\bar{s}_1)}{x'} \\
T_3^i &= - \frac{k_\perp'^i(\bar{s}_3)}{x'} + \frac{k_\perp'^i(s_3)}{(1 - x')} ; & T_4^i &= \frac{k_\perp^i(\bar{s}_1)}{(1 - x)} - \frac{k_\perp^i(s_1)}{x} \\
k_\perp^i(s) &= k_\perp^i + i s \varepsilon_{ij} k_\perp^j ; & \varepsilon_{ij} &= \varepsilon_{ij3} ; & \bar{s} &= -s
\end{aligned}$$

with the definitions

$$\begin{aligned}
\tilde{\Delta}_1 &= \frac{(xk'_\perp - x'k_\perp)^2 + m^2(x - x')^2}{xx'} ; & \tilde{\Delta}_2 &= \Delta_1|_{x \rightarrow (1-x), x' \rightarrow (1-x')} \\
\Delta_1 &= \frac{\tilde{\Delta}_1}{x' - x} ; & \Delta_2 &= \frac{\tilde{\Delta}_2}{x' - x} \\
\tilde{\Delta}_3 &= (k_\perp - k'_\perp)^2 + \frac{1}{2}(x - x')A + |x - x'| \left( \frac{1}{2}(M_0^2 + M_0'^2) - M_N^2 \right) \\
M_0^2 &= \frac{k_\perp^2 + m^2}{x(1 - x)} ; & M_0'^2 &= \frac{k_\perp'^2 + m^2}{x'(1 - x')} \\
A &= (k_\perp^2 + m^2) \left( \frac{1}{1 - x} - \frac{1}{x} \right) + (k_\perp'^2 + m^2) \left( \frac{1}{x'} - \frac{1}{1 - x'} \right) \\
P^- &= \frac{(P^\perp)^2 + M_N^2}{P^+} ; & P &= (P^+, P^\perp) ; & M_N &= 2m + B_N .
\end{aligned} \tag{136}$$

Note, that the rescaled potential, eq. (132), does not depend on the total momentum  $P^+$ , i.e. is invariant under light-front boosts.

The generated interaction plays an important role, namely it insures the absence of collinear divergencies in the renormalized interaction, which are associated with the limit  $x \rightarrow x'$ . This is true for any cutoff  $\lambda$ . In fact, in the limit when  $x$  tends to  $x'$  one has

$$\begin{aligned}
\tilde{\Delta}_1 &\sim \tilde{\Delta}_2 \longrightarrow (k_\perp - k'_\perp)^2 \\
\Delta_1 &\sim \Delta_2 \longrightarrow \frac{(k_\perp - k'_\perp)^2}{x' - x} .
\end{aligned} \tag{137}$$

This gives rise to

$$V_\lambda^{exch}(x \rightarrow x') = \left[ - \frac{e^2 N_{1,\lambda}}{(k_\perp - k'_\perp)^2} c_{ex}^{gen} - \frac{4e^2}{(x - x')^2} c_{ex}^{inst} \delta_{s_1 s_3} \delta_{s_2 s_4} \right] e^{-\left( \frac{M_0^2 - M_0'^2}{\lambda^2} \right)^2} , \tag{138}$$

where, in this limit, the perturbative part vanishes. Making use of eq. (135) we obtain for the divergent part  $\mathcal{DP}[\dots]$

$$\begin{aligned}
\mathcal{DP}[N_{1,\lambda}(x \rightarrow x')] &= T_1^\perp T_2^\perp \delta_{s_1 s_3} \delta_{s_2 s_4} \\
\mathcal{DP}[T_1^i(x \rightarrow x')] &= -2 \frac{(k_\perp - k'_\perp)^i}{(x - x')} ; & \mathcal{DP}[T_2^i(x \rightarrow x')] &= 2 \frac{(k_\perp - k'_\perp)^i}{(x - x')} .
\end{aligned} \tag{139}$$

This results in the exact cancellation of the collinear divergencies present initially in the instantaneous interaction by the generated term:

$$\mathcal{DP}[V_\lambda^{exch}(x \rightarrow x')] = 0 . \tag{140}$$



Note, that *no approximation* for the  $e\bar{e}$  interaction was made until now.

We rewrite both exchange and annihilation channel contributions in the **instant frame**

$$\begin{aligned}
V &= V_{exch} + V_{ann} \\
&= -e^2 N_{1,\lambda} \left[ \frac{1}{2} \left( \frac{1}{\Delta_1} + \frac{1}{\Delta_2} \right) (1 - e^{-2\frac{\Delta_1}{\lambda^2} \cdot \frac{\Delta_2}{\lambda^2}}) c_{ex}^{gen} + \frac{1}{\Delta_3} e^{-\left( \left( \frac{\Delta_1}{\lambda^2} \right)^2 + \left( \frac{\Delta_2}{\lambda^2} \right)^2 \right)} c_{ex}^{PT} \right] \\
&\quad + \left( -\frac{4e^2}{(x-x')^2} \delta_{s_1 s_3} \delta_{s_2 s_4} \right) c_{ex}^{inst} \\
&\quad + e^2 N_{2,\lambda} \left[ \frac{1}{M_0^2} \left( 1 - e^{-2\left( \frac{M_0^2}{\lambda^2} \right)^2} \right) c_{an}^{gen} + \frac{1}{M_N^2} e^{-2\left( \frac{M_0^2}{\lambda^2} \right)^2} c_{an}^{PT} \right] \\
&\quad + \left( 4e^2 \delta_{s_1 \bar{s}_3} \delta_{s_2 \bar{s}_4} \right) c_{an}^{inst}, \tag{141}
\end{aligned}$$

where quantities in the instant form are defined as follows

$$\begin{aligned}
x &= \frac{1}{2} \left( 1 + \frac{k_z}{\sqrt{p^2 + m^2}} \right); & x' &= \frac{1}{2} \left( 1 + \frac{k'_z}{\sqrt{p^2 + m^2}} \right) \\
M_0^2 &= M'^2 = 4(p^2 + m^2); \tag{142}
\end{aligned}$$

for the other quantities, defined in eqs. (135) and (136) the substitution  $x(k_z), x'(k'_z)$  is to be done. The symbols  $c_{ex}^{gen}$  and so on were introduced to indicate the origin of the different terms (here generated interaction coming from the exchange channel), all  $c = 1$ . We stress once more, that at the moment no approximation was done, therefore the expression of eq. (141) can be considered as the *exact electron-positron interaction, arising from the renormalized to the second order Hamiltonian*, that produces to the leading order the positronium mass  $M_N$  and the binding energy  $B_N$ . In fact, it will be shown, that the second order potential gives rise also to the correct mass splitting, that corresponds to the next to leading order contribution.

The expression of eq. (141) has a transparent form. First notice, that  $\Delta_1$  and  $\Delta_2$  describe the energy differences in two corresponding  $ee\gamma$ -vertices appearing in the  $e\bar{e}$  interaction. Then the generated interaction ( $c_{ex}^{gen}$ ) contributes mainly hard photon exchanges  $\frac{\Delta_1}{\lambda^2} \sim \frac{\Delta_2}{\lambda^2} \gg 1$ , while the term arising from perturbative theory  $c_{ex}^{PT}$  gives rise to soft photon exchanges. Though the renormalized interaction generally describes low energy physics, namely the renormalized Hamiltonian acts in the space where the energy differences are restricted by  $\lambda$ , the information on the high energy sector is accumulated in the generated interaction, making possible to interpolate between two sectors. This means, that the sum of both terms in eq. (141) recovers the *whole range of photon energies*. The same is true for the annihilation channel.

Two limits are of interest. The trivial limit  $\lambda \rightarrow \Lambda$  reproduces the bare interaction, used as initial condition for the renormalized interaction as  $\Lambda$  tends to infinity. In the opposite limit

$$\lambda \ll m \tag{143}$$

nonrelativistic physics is affected (see later). The limit  $\lambda \rightarrow 0$  can be performed *explicitly*, corresponding to the complete elimination of the  $ee\gamma$ -vertex present in the initial Hamiltonian. When  $\lambda$  tends to zero the  $e\bar{e}$  interaction is governed by generated and instantaneous terms. The fact that in this limit the potential is well defined is the direct consequence of the instant form frame.

The expression of eq. (141) for the renormalized to the second order interaction, written in the instant form frame, can be used as a suitable form to obtain the positronium spectrum

numerically. We further proceed in another direction, namely we make the nonrelativistic approximation (NR) for the electron (positron) momentum

$$\frac{|\vec{p}|}{m} = O(\alpha) < 1, \quad (144)$$

that enables to perform calculations for positronium mass splitting analitically.

#### 4.4 The nonrelativistic approximation

In the nonrelativistic approximation we obtain for the  $e\bar{e}$  interaction, eq. (141),

$$\begin{aligned} \tilde{V}_\lambda &= \frac{1}{2(2\pi)^3} \frac{1}{4m} \frac{1}{2m} \left(1 - \frac{\vec{p}^2}{2m^2}\right) V_\lambda \\ V_\lambda &= V_\lambda^{exch} + V_\lambda^{ann} \\ &= -\frac{e^2 N_1}{(\vec{p} - \vec{p}')^2} \left(1 - e^{-2\left(\frac{\Delta}{\lambda^2}\right)^2}\right) c_{ex}^{gen} - \frac{e^2 N_1}{(\vec{p} - \vec{p}')^2 + |x - x'| (M_0^2 - M_N^2)} e^{-2\left(\frac{\Delta}{\lambda^2}\right)^2} c_{ex}^{PT} \\ &\quad - \frac{4e^2}{(x - x')^2} \delta_{s_1 s_3} \delta_{s_2 s_4} c_{ex}^{inst} \\ &\quad + \frac{e^2 N_2}{4m^2} \left(1 - e^{-2\left(\frac{4m^2}{\lambda^2}\right)^2}\right) c_{an}^{gen} + \frac{e^2 N_2}{M_N^2} e^{-2\left(\frac{4m^2}{\lambda^2}\right)^2} c_{an}^{PT} \\ &\quad + 4e^2 \delta_{s_1 \bar{s}_3} \delta_{s_2 \bar{s}_4} c_{an}^{inst}, \end{aligned} \quad (145)$$

where the energy denominators and exponential factors were simplified using

$$\begin{aligned} x - x' &= \frac{k_z - k'_z}{2m} \left[1 + \frac{\vec{p}^2}{2m^2}\right] + O\left(m^2 \left(\frac{p}{m}\right)^5\right) \\ \tilde{\Delta}_1 &= \tilde{\Delta}_2 = (\vec{p} - \vec{p}')^2 + O\left(m^2 \left(\frac{p}{m}\right)^5\right) \\ \tilde{\Delta}_3 &= (\vec{p} - \vec{p}')^2 + |x - x'| (M_0^2 - M_N^2) + O\left(m^2 \left(\frac{p}{m}\right)^4\right) \\ \Delta_1 &= \Delta_2 = \frac{2m(\vec{p}' - \vec{p})^2}{(k'_z - k_z)} \left[1 + O\left(\left(\frac{p}{m}\right)^2\right)\right]; \quad \Delta = \frac{2m(\vec{p}' - \vec{p})^2}{(k'_z - k_z)} \\ M_0^2 &= 4m^2 + O\left(m^2 \left(\frac{p}{m}\right)^2\right) \\ M_N^2 &= 4m^2 + 4mB_N + O\left(m^2 \left(\frac{B_N}{m}\right)^2\right) = 4m^2 + 4mB_N^{(0)}, \end{aligned} \quad (146)$$

and the explicit expression of Jacobian for the coordinate change is

$$\sqrt{J(p)J(p')} = \frac{1}{2m} \left[1 - \frac{\vec{p}^2}{2m^2} + O\left(\frac{k_z^2}{m^2}, \frac{k_z'^2}{m^2}\right)\right], \quad (147)$$

having introduced the leading order binding energy  $B_N^{(0)}$ . Making use of its nonrelativistic approximation  $B_N^{(0)}/m \ll 1$  we have for the interaction

$$\tilde{V}_\lambda = \frac{1}{2(2\pi)^3} \frac{1}{4m} \frac{1}{2m} \left(1 - \frac{\vec{p}^2}{2m^2}\right)$$

$$\begin{aligned}
& \times \left[ \left( -\frac{e^2 N_1}{(\vec{p} - \vec{p}')^2} (c_{ex}^{gen}, c_{ex}^{PT}) - \frac{16e^2 m^2}{(k_z - k'_z)^2} \left( 1 + \frac{\vec{p}^2}{m^2} \right) c_{ex}^{inst} \delta_{s_1 s_3} \delta_{s_2 s_4} \right. \right. \\
& \quad \left. \left. + \frac{e^2 N_2}{4m^2} (c_{an}^{gen}, c_{an}^{PT}) + 4e^2 c_{an}^{inst} \delta_{s_1 \bar{s}_2} \delta_{s_3 \bar{s}_4} \right) \right. \\
& \quad \left. + \left( -\frac{e^2 N_1}{(\vec{p} - \vec{p}')^2} \frac{4m B_N^{(0)}}{|\Delta|} c_{ex}^{PT} e^{-\left(\frac{\Delta}{\lambda^2}\right)^2} \right. \right. \\
& \quad \left. \left. - \frac{e^2 N_2}{4m^2} \frac{B_N^{(0)}}{m} c_{an}^{PT} e^{-\left(\frac{4m^2}{\lambda^2}\right)^2} \right) \right], \tag{148}
\end{aligned}$$

where  $(c^{gen}, c^{PT})$  shows that both term, generated and perturbative interactions, contribute to corresponding term (we remember that all  $c = 1$ ).

The remarkable feature of the part of interaction standing in the first bracket is that it does not depend on the UV cutoff  $\lambda$ . The next term in the second bracket arises from the perturbative photon exchange and has the typical 'energy shell' structure for the relativistic effects, namely these terms are important when  $\lambda \gg m$ . Further we calculate the ground state positronium mass and therefore restrict the cutoff to be in the nonrelativistic domain

$$\lambda \ll m, \tag{149}$$

where the second term in eq. (148) vanishes and we are left with the following form for the renormalized  $e\bar{e}$  interaction in the nonrelativistic approximation:

$$\begin{aligned}
\tilde{V}(\vec{p}, \vec{p}') &= \frac{1}{2(2\pi)^3} \frac{1}{4m} \frac{1}{2m} \left( 1 - \frac{\vec{p}^2}{2m^2} \right) \\
&\times \left[ -\frac{e^2 N_1}{(\vec{p} - \vec{p}')^2} (c_{ex}^{eff}, c_{ex}^{PT}) - \frac{16e^2 m^2}{(k_z - k'_z)^2} \left( 1 + \frac{\vec{p}^2}{m^2} \right) c_{ex}^{inst} \delta_{s_1 s_3} \delta_{s_2 s_4} \right. \\
&\quad \left. + \frac{e^2 N_2}{4m^2} (c_{an}^{eff}, c_{an}^{PT}) + 4e^2 c_{an}^{inst} \delta_{s_1 \bar{s}_2} \delta_{s_3 \bar{s}_4} \right]. \tag{150}
\end{aligned}$$

We perform the nonrelativistic expansion of the factors  $N_1$  and  $N_2$  appearing in the interaction. The term  $N_1$  contributes in  $\tilde{V}$  to the order

$$\underline{O(1), O\left(\left(\frac{p}{m}\right)^2\right)}:$$

$$\begin{aligned}
-T_1^\perp T_2^\perp &= 16m^2 \frac{q_\perp^2}{q_z^2} \left( 1 + \frac{\vec{p}^2}{m^2} \right) + 16 \frac{q_\perp^i}{q_z} (k_\perp^i k_z + k_\perp'^i k_z') \\
&\quad - 16i(s_1 + s_2)[k_\perp' k_\perp] - 4(k_\perp + k_\perp')^2 + 4s_1 s_2 q_\perp^2,
\end{aligned}$$

$$\underline{O\left(\frac{p}{m}\right), O\left(\left(\frac{p}{m}\right)^2\right)}:$$

$$\begin{aligned}
& im\sqrt{2}(x' - x) \left( \frac{s_1}{xx'} \varepsilon_{s_1}^\perp \cdot T_1^\perp \delta_{\bar{s}_1 s_3} \delta_{s_2 s_4} + \frac{s_2}{(1-x)(1-x')} \varepsilon_{s_2}^\perp \cdot T_2^\perp \delta_{\bar{s}_4 s_2} \delta_{s_1 s_3} \right) \\
&= 8 \delta_{\bar{s}_1 s_3} \delta_{s_2 s_4} \left[ m(iq_\perp^x - s_1 q_\perp^y) \left( 1 - \frac{k_z + k'_z}{m} \right) + q_z(i\tilde{p}_\perp^x - s_1 \tilde{p}_\perp^y) + \frac{1}{2} s_2 q_z(q_\perp^y - i s_1 q_\perp^x) \right] \\
&\quad - \delta_{\bar{s}_4 s_2} \delta_{s_1 s_3} \left[ m(iq_\perp^x - s_2 q_\perp^y) \left( 1 + \frac{k_z + k'_z}{m} \right) - q_z(i\tilde{p}_\perp^x - s_2 \tilde{p}_\perp^y) - \frac{1}{2} s_1 q_z(q_\perp^y - i s_2 q_\perp^x) \right],
\end{aligned}$$

$$\underline{O\left(\left(\frac{p}{m}\right)^2\right)} :$$

$$2m^2 \frac{(x-x')^2}{xx'(1-x)(1-x')} = 8q_z^2 .$$

Whereas the term  $N_2$  contributes to  $\tilde{V}$  to the order

$$\underline{O\left(\left(\frac{p}{m}\right)^2\right)} :$$

$$2m^2 \frac{1}{xx'(1-x)(1-x')} = 32m^2 . \quad (151)$$

In these formulas we have used  $[k'_\perp, k_\perp] = \varepsilon_{ij} k'^i_\perp k^j_\perp$ ,  $\varepsilon_{ij} = \varepsilon_{ij3}$  and  $\varepsilon^i_s = -\frac{1}{\sqrt{2}}(s, i)$ ; also the following variables have been introduced

$$\begin{aligned} q_\perp &= k'_\perp - k_\perp , \quad (\perp = x, y) , \quad q_z = k'_z - k_z \\ \tilde{p}_\perp &= \frac{k_\perp + k'_\perp}{2} . \end{aligned} \quad (152)$$

We leave aside for the future work the analysis of the expressions for  $N_1$  and  $N_2$ , where also in this form some terms can be identified as spin-orbit and spin-spin interactions in the transverse plane and in longitudinal (z) direction.

Instead we follow [6], where an analogous calculation of singlet-triplet ground state mass splitting of positronium was performed in the similarity scheme. This means, that we can, except for the leading order term  $O(1)$ , drop in  $N_1$  the part diagonal in spin space. Also the terms of the type  $f = k^{x,y}_\perp k_z$ ,  $k^{x,y}_\perp k'_z$ ,  $k^x_\perp k^y_\perp$  do not contribute to the ground state mass splitting, since

$$\int d^3p d^3p' \Phi_{100}^*(\vec{p}) \frac{f}{\vec{q}^2} \Phi_{100}(\vec{p}') , \quad (153)$$

averaging over directions, gives zero.

We obtain for the  $e\bar{e}$ -potential to the leading order  $O(1)$  of nonrelativistic expansion

$$\begin{aligned} \tilde{V}^{(0)}(\vec{p}', \vec{p}) &= \frac{1}{2(2\pi)^3} \frac{1}{4m} \frac{1}{2m} \left( 1 - \frac{\vec{p}^2}{2m^2} \right) \\ &\times \left[ \frac{16e^2 m^2}{\vec{q}^2} \frac{q_\perp^2}{q_z^2} \left( 1 + \frac{\vec{p}^2}{m^2} \right) (c_{ex}^{gen}, c_{ex}^{PT}) - \frac{16e^2 m^2}{q_z^2} \left( 1 + \frac{\vec{p}^2}{m^2} \right) c_{ex}^{inst} \right] \delta_{s_1 s_3} \delta_{s_2 s_4} \\ &= -\frac{\alpha}{2\pi^2} \frac{1}{\vec{q}^2} \left( 1 + \frac{\vec{p}^2}{2m^2} \right) \delta_{s_1 s_3} \delta_{s_2 s_4} \\ &\longrightarrow V(r) \left( 1 + \frac{\vec{p}^2}{2m^2} \right) . \end{aligned} \quad (154)$$

Remembering  $\vec{q} = \vec{p}' - \vec{p}$ , Fourier transformation to the coordinate space with respect to  $\vec{q}$  has been performed in the last expression. To the leading order of NR expansion we have reproduced the Coulomb potential, defined before as the leading order of BSPT. Note, this is true for any UV cutoff within the nonrelativistic range  $\lambda \ll m$ .

We combine this expression with the kinetic term from the Schrödinger equation, eq. (125), and write it in the form

$$\frac{1}{m} \left( 1 + \frac{V(r)}{2m} \right) \vec{p}^2 + V(r) . \quad (155)$$

Here the potential  $V(r)$  plays a different role in the two terms. In the first term, corresponding to kinetic energy, it generates an effective mass of the electron, which depends on the relative position and manifests the non-locality of the interaction. The second term is the usual potential energy, in our case, the Coulomb interaction.

The energy of the Coulomb level with quantum numbers  $(nlm)$  is standard

$$M_0^2 = \langle \Phi_{nlm} | \tilde{V}^{(0)} | \Phi_{nlm} \rangle = \int d^3p d^3p' \Phi_{nlm}^*(\vec{p}) \tilde{V}^{(0)} \Phi_{nlm}(\vec{p}') = -\frac{m\alpha^2}{2n^2}, \quad (156)$$

where the Coulomb wave functions  $\Phi_{nlm}$  were defined in eq. (127). We have used in eq. (156) the following representation

$$\begin{aligned} (\vec{p} - \vec{p}')^2 &= \frac{(e_n^2 + \vec{p}^2)(e_n^2 + \vec{p}'^2)}{4e_n^2} (u - u')^2 \\ \frac{1}{(u - u')^2} &= \sum_{\mu} \frac{2\pi^2}{n} Y_{\mu}(\Omega_p) Y_{\mu}^*(\Omega_{p'}) \\ d^3p &= d\Omega_p \left( \frac{e_n^2 + \vec{p}^2}{2e_n} \right)^3 \end{aligned} \quad (157)$$

and also orthogonality of the hyperspherical harmonics

$$\int d\Omega Y_{\mu}^* Y_{\mu'} = \delta_{\mu\mu'}. \quad (158)$$

More details can be found in [6].

The next to leading order  $O\left(\frac{p}{m}\right)$

$$\begin{aligned} \delta V^{(1)} &= \frac{1}{2(2\pi)^3} \frac{1}{4m} \frac{1}{2m} \left( -\frac{e^2}{\vec{q}^2} \right) \\ &\times (8m(iq_{\perp}^x - s_1 q_{\perp}^y) \delta_{s_1 \bar{s}_3} \delta_{s_2 s_4} - 8m(iq_{\perp}^x - s_2 q_{\perp}^y) \delta_{s_1 s_3} \delta_{s_2 \bar{s}_4}) \end{aligned} \quad (159)$$

contributes (because of the spin structure) to the second order of BSPT:

$$\delta M_2^2 = \sum_{\mu, s_i} \frac{\langle \Phi_{100} | \delta V^{(1)} | \Phi_{\mu, s_i} \rangle \langle \Phi_{\mu, s_i} | \delta V^{(1)} | \Phi_{100} \rangle}{M_1^2 - M_n^2}. \quad (160)$$

The order  $O\left(\left(\frac{p}{m}\right)^2\right)$  (cf. remark after eq. (152)) is

$$\delta V^{(2)} = \frac{1}{2(2\pi)^3} \frac{1}{4m} \frac{1}{2m} \left( 8e^2 \frac{q_z^2}{\vec{q}^2} \delta_{s_1 \bar{s}_2} \delta_{s_1 \bar{s}_3} \delta_{s_2 \bar{s}_4} + 8e^2 \delta_{s_1 s_2} \delta_{s_3 s_4} \delta_{s_1 s_3} + 4e^2 \delta_{s_1 \bar{s}_3} \delta_{s_2 \bar{s}_4} \right) \quad (161)$$

and contributes to the first order of BSPT:

$$\delta M_1^2 = \langle \Phi_{100} | \delta V^{(2)} | \Phi_{100} \rangle. \quad (162)$$

Both contributions were calculated in [6] with the result

$$\begin{aligned} \delta M^2 &= \delta M_1^2 + \delta M_2^2 \\ \langle 1 | \delta M^2 | 1 \rangle &= -\frac{5}{12} m\alpha^4 \\ \langle 2 | \delta M^2 | 2 \rangle &= \langle 3 | \delta M^2 | 3 \rangle = \langle 4 | \delta M^2 | 4 \rangle = \frac{1}{6} m\alpha^4, \end{aligned} \quad (163)$$

where the eigenvectors in spin space are defined as follows:

$$\begin{aligned} |1\rangle &= \frac{1}{\sqrt{2}} (|+- \rangle - |-+\rangle) , \\ |2\rangle &= \frac{1}{\sqrt{2}} (|+- \rangle + |-+\rangle) , \quad |3\rangle = |- - \rangle , \quad |4\rangle = |+ + \rangle . \end{aligned} \quad (164)$$

Making use of the relation between Coulomb energy units and  $\text{Ryd} = \frac{1}{2}m\alpha^2$  we have the standard result for the singlet-triplet mass splitting for positronium,  $\frac{7}{6}\alpha^2\text{Ryd}$ . The degeneracy of the triplet ground state  $n = 1$  reflects the rotational invariance, manifest in the system in nonrelativistic approximation.

## 5 Conclusion

Basing on the similarity scheme together with the flow equation method we derive the renormalized to the second order  $O(e^2)$  LFQED Hamiltonian. It has, in its turn, two aspects considered further in the work.

The first point is related with renormalization group. It turns out, that electron (photon) mass correction and wave function renormalization constant vary with UV cutoff in accordance to 1-loop renormalization group equations. This indicates an intimate connection between Wilson's renormalization and the flow equation method. The same equivalence to Wilson's renormalization was shown for the projection operator techniques (Bloch-Feshbach formalism) [7].

In the case of LFQED two types of divergencies arise: UV divergencies, associated with a large transverse momentum and severe IR divergencies, originating from light-front gauge singularities and present in light-front Hamiltonian calculations for QED. Flow equation method deliver in divergent integrals straightforwardly the regulator, that regularize UV and partially IR divergencies. It was shown in the work, that gauge invariant calculations of divergent terms give rise to IR finite results. Explicitly, the IR divergencies were removed from the electron (photon) energy correction, when all diagrams to the second order in the corresponding sectors (arising from flow equations and normal-ordering Hamiltonian) were considered. This enables to choose IR-independent mass counterterms and insures IR-finite physical masses.

To complete the renormalization group analysis the third order (coupling constant renormalization) and Ward identities must be considered in the techniques discussed.

The second aspect is connected with the low-energy sector of the theory and bound state calculations. The renormalized Hamiltonian acts in the space where the energy differences between initial and final states are restricted by the UV cutoff, and therefore it can be interpreted as an effective Hamiltonian descibing the low-energy physics.

Making use of diagrammatic rules for the renormalized LFQED Hamiltonian, we obtain the electron-positron interaction to the second order. The generated interaction, arising from elimination of  $ee\gamma$ -vertex, insures at any cutoff  $\lambda$  the absence of collinear divergencies in the  $e\bar{e}$ -interaction.

The  $e\bar{e}$ -interaction possesses the explicit and implicit, through the running mass and coupling, cutoff dependence. We aimed in the work to get rid of explicit  $\lambda$ -dependence, performing the limit  $\lambda \rightarrow 0$  or making use of the nonrelativistic approximation, to obtain the physical results for the spectrum. The instant form frame was used, that manifest several advantages as compared with the light-front one. First, the complete elimination of the  $ee\gamma$ -vertex is possible, that corresponds to the limit  $\lambda \rightarrow 0$  where the  $e\bar{e}$ -interaction obviously does not depend any

more on the cutoff  $\lambda$ . In this limit the interaction is governed by generated and instantaneous terms. Note, the light-front bound state equation is boost and frame invariant, giving rise to the same spectrum in both frames.

To calculate the positronium spectrum analytically the nonrelativistic approximation with respect to the electron momentum  $\frac{|\vec{p}|}{m} = O(\alpha)$  and binding energy  $\frac{B_N}{m} = O(\alpha)$  was performed. The instant form enables then to obtain the cutoff independent  $e\bar{e}$ -interaction as long as the UV cutoff is restricted to be in the nonrelativistic domain, namely  $\lambda \ll m$ . The leading order (in nonrelativistic expansion) of this interaction describes 3d Coulomb interaction, giving rise to standard Coulomb energy levels and wave functions. Note, that in the nonrelativistic approximation it occurs exact cancellation of instantaneous interaction by the leading order generated and perturbative theory terms, giving rise to the 3d-Coulomb interaction. The next order terms define the spin structure of the  $e\bar{e}$ -interaction. The same result for the spin-dependent terms was obtained in [6], where the ground state singlet-triplet mass splitting for positronium was calculated to be  $\frac{7}{6}\alpha^2 Ryd$ . The mass degeneracy of the triplet ground state manifests the rotational invariance, maintained at the order  $\alpha^4$ .

The correct results obtained for the positronium mass spectrum in nonrelativistic approximation are encouraging. This makes possible to consider an exact expression for the  $e\bar{e}$ -interaction, written in instant form frame, as a basis for future numerical calculations.

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# A Renormalization scheme of similarity unitary transformation

Here we discuss briefly the renormalization scheme presented by Glazek and Wilson [2], called similarity unitary transformation, and compare it with the flow equations by Wegner.

Both approaches aim to bring the Hamiltonian to the most diagonal form, explicitly only the matrix elements between the free states with

$$|\Delta_{ij}| < \lambda \quad (165)$$

and  $\Delta_{ij} = E_i - E_j$ , are present in the renormalized Hamiltonian. For this purposes the continuous unitary transformation must be performed to preserve unchanged the spectrum (eigenvalues) of the initial bare cutoff Hamiltonian. The demand of diagonal structure does not define completely the generator of the transformation. This freedom is used to eliminate small energy denominators in the final renormalized Hamiltonian. This results in a system of two self-consistent non-linear differential equations for the Hamiltonian  $H(l)$  and the generator of the transformation  $\eta(l)$ . The dependence on the continuous flow parameter  $l$  in the flow equations by Wegner is replaced by the cutoff dependence  $\lambda$  in the similarity approach, with the connection

$$l = 1/\lambda^2 \quad (166)$$

in the renormalized Hamiltonians.

The difference of the two methods manifests the residual freedom in the choice of the direction of the infinitesimal rotation, actually defining how fast the non-diagonal matrix elements vanish.

We summarize the equations for both schemes, written in matrix form. Remember, the function  $f_{ij}$  defines the solution for the leading order interaction term.

**I.** The flow equations by Wegner [1]:

$$\frac{dH_{ij}}{dl} = [\eta, H_I]_{ij} + \frac{du_{ij}}{dl} \frac{H_{ij}}{u_{ij}}, \quad (167)$$

$$\eta_{ij} = \frac{1}{E_i - E_j} \left( -\frac{du_{ij}}{dl} \frac{H_{ij}}{u_{ij}} \right) \quad (168)$$

with

$$u_{ij} = \exp(-l\Delta_{ij}^2) \quad (169)$$

and

$$f_{ij} = u_{ij}. \quad (170)$$

**II.** The similarity unitary transformation by Glazek and Wilson [2]:

$$\frac{dH_{ij}}{d\lambda} = u_{ij}[\eta, H_I]_{ij} + r_{ij} \frac{du_{ij}}{d\lambda} \frac{H_{ij}}{u_{ij}}, \quad (171)$$

$$\eta_{ij} = \frac{r_{ij}}{E_i - E_j} \left( [\eta, H_I]_{ij} - \frac{du_{ij}}{d\lambda} \frac{H_{ij}}{u_{ij}} \right) \quad (172)$$

and

$$f_{ij} = u_{ij} \exp(r_{ij}). \quad (173)$$



Also the following transformation is used [2]:

$$\frac{dH_{ij}}{d\lambda} = u_{ij}[\eta, H_I]_{ij} + \frac{du_{ij}}{d\lambda} \frac{H_{ij}}{u_{ij}}, \quad (174)$$

$$\eta_{ij} = \frac{1}{E_i - E_j} \left( r_{ij}[\eta, H_I]_{ij} - \frac{du_{ij}}{d\lambda} \frac{H_{ij}}{u_{ij}} \right) \quad (175)$$

and

$$f_{ij} = u_{ij}, \quad (176)$$

where for both similarity transformations

$$u_{ij} = \theta(\lambda - |\Delta_{ij}|) \quad (177)$$

and

$$u_{ij} + r_{ij} = 1. \quad (178)$$

Also other choices for the similarity function  $u_{ij}$  with the step behaviour are possible [2].

Remember, the function  $f_{ij}$  defines the solution for the leading order interaction term. The first order equations for  $H$  and  $\eta$ , written through the  $f$ -function are unique for both methods (**I.** and **II.**)

$$H_{I,ij}^{(1)}(l) = H_{I,ij}^{(1)}(l=0) \frac{f_{ij}(l)}{f_{ij}(l=0)}, \quad (179)$$

$$\eta_{ij}^{(1)}(l) = -\frac{1}{E_i - E_j} \frac{dH_{I,ij}^{(1)}}{dl} \quad (180)$$

with the connection given in eq. (166) in the renormalized values, and  $dl \rightarrow d\lambda$  implied. This will be exploited further for the calculations in the main text.

## B Calculation of $[\eta^{(1)}(l), H_{ee\gamma}]$ in the $e\bar{e}$ -sector

Here we calculate the commutator  $[\eta^{(1)}(l), H_{ee\gamma}]$  in the electron-positron sector. The leading order generator  $\eta^{(1)}$  is:

$$\begin{aligned} \eta^{(1)}(l) = & \sum_{\lambda s_1 s_3} \int_{p_1 p_3 q} (\eta_{p_1 p_3}^*(l) \varepsilon_{\lambda}^i \tilde{a}_q + \eta_{p_1 p_3}(l) \varepsilon_{\lambda}^{i*} \tilde{a}_{-q}^+) (\tilde{b}_{p_3}^+ \tilde{b}_{p_1} + \tilde{b}_{p_3}^+ \tilde{d}_{-p_1}^+ + \tilde{d}_{-p_3} \tilde{b}_{p_1} + \tilde{d}_{-p_3} \tilde{d}_{-p_1}^+) \\ & \times \chi_{s_3}^+ \Gamma_l^i(p_1, p_3, -q) \chi_{s_1} \delta_{q, -(p_1 - p_3)}, \end{aligned} \quad (181)$$

where

$$\eta_{p_1 p_3}(l) = -\Delta_{p_1 p_3} \cdot g_{p_1 p_3} = \frac{1}{\Delta_{p_1 p_3}} \cdot \frac{dg_{p_1 p_3}}{dl}, \quad (182)$$

$\Delta_{p_1 p_3} = p_1^- - p_3^- - (p_1 - p_3)^-$ , and the electron-photon coupling

$$\begin{aligned} H_{ee\gamma} = & \sum_{\lambda s_2 s_4} \int_{p_2 p_4 q'} (g_{p_2 p_4}^*(l) \varepsilon_{\lambda}^j \tilde{a}_{q'} + g_{p_2 p_4}(l) \varepsilon_{\lambda}^{j*} \tilde{a}_{-q'}^+) (\tilde{b}_{p_4}^+ \tilde{b}_{p_2} + \tilde{b}_{p_4}^+ \tilde{d}_{-p_2}^+ + \tilde{d}_{-p_4} \tilde{b}_{p_2} + \tilde{d}_{-p_4} \tilde{d}_{-p_2}^+) \\ & \times \chi_{s_4}^+ \Gamma_l^j(p_2, p_4, -q') \chi_{s_2} \delta_{q', -(p_2 - p_4)}, \end{aligned} \quad (183)$$

where

$$\Gamma_l^i(p_1, p_2, q) = 2 \frac{q^i}{q^+} - \frac{\sigma \cdot p_2^\perp - im}{p_2^+} \sigma^i - \sigma^i \frac{\sigma \cdot p_1^\perp + im}{p_1^+} \quad (184)$$

and the tilde-fields are defined in eq. (33). Further we use the identities for the polarisation vectors and spinors

$$\sum_{\lambda} \varepsilon_{\lambda}^{i*} \varepsilon_{\lambda}^j = \delta^{ij} , \quad \chi_s^+ \chi_{s'} = \delta_{ss'} . \quad (185)$$

Using the commutation relations, eq. (25), and identities eq. (185) we have

$$\begin{aligned} [\eta^{(1)}(l), H_{ee\gamma}] &= \frac{1}{2} \left( -\eta_{p_1 p_3} g_{p_2 p_4}^* \frac{\theta(p_1^+ - p_3^+)}{p_1^+ - p_3^+} + \eta_{p_1 p_3}^* g_{p_2 p_4} \frac{\theta(p_3^+ - p_1^+)}{p_3^+ - p_1^+} \right) \\ &\times : (-\tilde{b}_{p_3}^+ \tilde{d}_{-p_2}^+ \tilde{d}_{-p_4} \tilde{b}_{p_1} - \tilde{b}_{p_4}^+ \tilde{d}_{-p_1}^+ \tilde{d}_{-p_3} \tilde{b}_{p_2} + \tilde{b}_{p_3}^+ \tilde{d}_{-p_1}^+ \tilde{d}_{-p_4} \tilde{b}_{p_2} + \tilde{b}_{p_4}^+ \tilde{d}_{-p_2}^+ \tilde{d}_{-p_3} \tilde{b}_{p_1}) : \\ &\times (\chi_{s_3}^+ \Gamma_l^i(p_1, p_3, p_1 - p_3) \chi_{s_1}) (\chi_{s_4}^+ \Gamma_l^i(p_2, p_4, p_2 - p_4) \chi_{s_2}) \delta_{p_1+p_2, p_3+p_4} , \end{aligned} \quad (186)$$

where the first two terms of the field operators contribute to the exchange channel, and the next two to the annihilation channel. We take into account both  $s$ - and  $t$ -channel terms to calculate the bound states. The  $:$  stand for the normal ordering of the fermion operators and  $(\frac{1}{2})$  is the symmetry factor. The sum over helicities  $s_i$  and the 3-dimensional integration over momenta  $p_i$ ,  $i = 1, \dots, 4$ , according to eq. (34) is implied. We rewrite for both channels

$$[\eta, H_{ee\gamma}] = \begin{cases} M_{2ij}^{(ex)}(\frac{1}{2}) \left\{ \frac{\theta(p_1^+ - p_3^+)}{(p_1^+ - p_3^+)} (\eta_{p_1, p_3} g_{-p_4, -p_2}^* - \eta_{-p_4, -p_2}^* g_{p_1, p_3}) \right. \\ \quad \left. + \frac{\theta(-(p_1^+ - p_3^+))}{-(p_1^+ - p_3^+)} (\eta_{-p_4, -p_2} g_{p_1, p_3}^* - \eta_{p_1, p_3}^* g_{-p_4, -p_2}) \right\} \\ \quad \times \delta^{ij} \delta_{p_1+p_2, p_3+p_4} b_{p_3 s_3}^+ d_{p_4 \bar{s}_4}^+ d_{p_2 \bar{s}_2} b_{p_1 s_1} \\ \\ -M_{2ij}^{(an)}(\frac{1}{2}) \left\{ \frac{\theta(p_1^+ + p_2^+)}{(p_1^+ + p_2^+)} (\eta_{p_1, -p_2} g_{-p_4, p_3}^* - \eta_{-p_4, p_3}^* g_{p_1, -p_2}) \right. \\ \quad \left. + \frac{\theta(-(p_1^+ + p_2^+))}{-(p_1^+ + p_2^+)} (\eta_{-p_4, p_3} g_{p_1, -p_2}^* - \eta_{p_1, -p_2}^* g_{-p_4, p_3}) \right\} \\ \quad \times \delta^{ij} \delta_{p_1+p_2, p_3+p_4} b_{p_3 s_3}^+ d_{p_4 \bar{s}_4}^+ d_{p_2 \bar{s}_2} b_{p_1 s_1} \end{cases} \quad (187)$$

where

$$\begin{aligned} M_{2ij}^{(ex)} &= (\chi_{s_3}^+ \Gamma_l^i(p_1, p_3, p_1 - p_3) \chi_{s_1}) (\chi_{s_2}^+ \Gamma_l^j(-p_4, -p_2, -(p_1 - p_3)) \chi_{\bar{s}_4}) \\ M_{2ij}^{(an)} &= (\chi_{s_3}^+ \Gamma_l^i(-p_4, p_3, -(p_1 + p_2)) \chi_{\bar{s}_4}) (\chi_{s_2}^+ \Gamma_l^j(p_1, -p_2, p_1 + p_2) \chi_{s_1}) . \end{aligned} \quad (188)$$

The first term in the exchange channel with  $p_1^+ > p_3^+$  corresponds to the light-front time ordering  $x_1^+ < x_3^+$  with the intermediate state  $P_k^- = p_3^- + (p_1 - p_3)^- + p_2^-$ , the second term  $p_1^+ < p_3^+$  and  $x_1^+ > x_3^+$  has the intermediate state  $P_k^- = p_1^- - (p_1 - p_3)^- + p_4^-$ . Both terms can be viewed as the retarded photon exchange. The same does hold for the annihilation channel.

Consider only real couplings and take into account the symmetry

$$\eta_{-p_4, -p_2} = -\eta_{p_4, p_2} , \quad g_{-p_4, -p_2} = g_{p_4, p_2} . \quad (189)$$

Then  $\langle p_3 s_3, p_4 \bar{s}_4 | [\eta^{(1)}, H_{ee\gamma}] | p_1 s_1, p_2 \bar{s}_2 \rangle$ , the matrix element of the commutator between the free states of positronium in the exchange and annihilation channel, reads

$$\langle [\eta^{(1)}, H_{ee\gamma}] \rangle / \delta_{p_1+p_2, p_3+p_4} = \begin{cases} M_{2ii}^{ex} \frac{1}{(p_1^+ - p_3^+)} (\eta_{p_1, p_3} g_{p_4, p_2} + \eta_{p_4, p_2} g_{p_1, p_3}) \\ -M_{2ii}^{an} \frac{1}{(p_1^+ + p_2^+)} (\eta_{p_1, -p_2} g_{p_4, -p_3} + \eta_{p_4, -p_3} g_{p_1, -p_2}) \end{cases} . \quad (190)$$

We rewrite this expression through the corresponding  $f$ -functions

$$\eta_{p_1,p_3}g_{p_4,p_2} + \eta_{p_4,p_2}g_{p_1,p_3} = e^2 \left[ \frac{1}{\Delta_{p_1,p_3}} \frac{df_{p_1,p_3}(l)}{dl} f_{p_4,p_2}(l) + \frac{1}{\Delta_{p_4,p_2}} \frac{df_{p_4,p_2}(l)}{dl} f_{p_1,p_3}(l) \right] \quad (191)$$

$$\eta_{p_1,-p_2}g_{p_4,-p_3} + \eta_{p_4,-p_3}g_{p_1,-p_2} = e^2 \left[ \frac{1}{\Delta_{p_1,-p_2}} \frac{df_{p_1,-p_2}(l)}{dl} f_{p_4,-p_3}(l) + \frac{1}{\Delta_{p_4,-p_3}} \frac{df_{p_4,-p_3}(l)}{dl} f_{p_1,-p_2}(l) \right]$$

with  $\Delta_{p_1,p_2} = p_1^- - p_2^- - (p_1 - p_2)^-$ . As we have mentioned in Appendix A this form in terms of the  $f$ -function is universal for all unitary transformations. We exploit further this expression by specifying the  $f$ -function to compare the effective interactions in different renormalization schemes (see Appendix C).

We calculate the matrix elements  $M_{2ii}$ , eq. (188), for both channels. Here we follow the notations introduced in [6].

We make use of the identities

$$\chi_s^+ \sigma^i \sigma^j \chi_s = \delta^{ij} + i s \varepsilon^{ij}, \quad \chi_s^+ \sigma^j \sigma^i \chi_s = \delta^{ij} + i \bar{s} \varepsilon^{ij} \quad (192)$$

with  $\bar{s} = -s$  and  $\chi_s^+ \chi_{s'} = \delta_{ss'}$ ; also of

$$\chi_{\bar{s}}^+ \sigma^i \chi_s = -\sqrt{2} s \varepsilon_s^i, \quad \chi_s^+ \sigma^i \chi_{\bar{s}} = -\sqrt{2} s \varepsilon_s^{i*} \quad (193)$$

with  $\varepsilon_s^* = -\varepsilon_{\bar{s}}$  and  $\varepsilon_s^i \varepsilon_{s'}^i = -\delta_{ss'}$ .

We use the standard light-front frame, fig. (4),

$$\begin{aligned} p_1 &= (xP^+, xP^\perp + k_\perp), & p_2 &= ((1-x)P^+, (1-x)P^\perp - k_\perp), \\ p_3 &= (x'P^+, x'P^\perp + k'_\perp), & p_4 &= ((1-x')P^+, (1-x')P^\perp - k'_\perp), \end{aligned} \quad (194)$$

where  $P = (P^+, P^\perp)$  is the positronium momentum.

Then, to calculate the matrix element  $M_{2ii}$  in the **exchange channel**, we find

$$\begin{aligned} P^+ [\chi_{s_3}^+ \Gamma^i(p_1, p_3, p_1 - p_3) \chi_{s_1}] &= \chi_{s_3}^+ \left[ 2 \frac{(k_\perp - k'_\perp)^i}{(x - x')} - \frac{\sigma \cdot k'_\perp}{x'} \sigma^i + \sigma^i \frac{\sigma \cdot k_\perp}{x} + im \frac{x - x'}{xx'} \sigma^i \right] \chi_{s_1} \\ &= T_2^i \delta_{s_1 s_3} + im \frac{x - x'}{xx'} (-\sqrt{2}) s_1 \varepsilon_{s_1}^i \delta_{s_1 \bar{s}_3}, \end{aligned} \quad (195)$$

and

$$\begin{aligned} P^+ [\chi_{\bar{s}_2}^+ \Gamma^i(-p_4, -p_2, -(p_4 - p_2)) \chi_{\bar{s}_4}] &= \chi_{\bar{s}_2}^+ \left[ 2 \frac{(k_\perp - k'_\perp)^i}{x - x'} + \left( \frac{\sigma \cdot k_\perp}{1-x} \sigma^i + \sigma^i \frac{\sigma \cdot k'_\perp}{1-x'} \right) - im \frac{x - x'}{(1-x)(1-x')} \sigma^i \right] \chi_{\bar{s}_4} \\ &= - \left[ T_1^i \delta_{s_2 s_4} + im \frac{x - x'}{(1-x)(1-x')} (-\sqrt{2}) s_2 \varepsilon_{s_2}^i \delta_{s_2 \bar{s}_4} \right], \end{aligned} \quad (196)$$

where we have introduced

$$\begin{aligned} T_1^i &\equiv - \left[ 2 \frac{(k_\perp - k'_\perp)^i}{x - x'} + \frac{k_\perp^i(s_2)}{(1-x)} + \frac{k'^i_\perp(\bar{s}_2)}{(1-x')} \right] \\ T_2^i &\equiv 2 \frac{(k_\perp - k'_\perp)^i}{x - x'} - \frac{k_\perp^i(s_1)}{x} - \frac{k'^i_\perp(\bar{s}_1)}{x'} \end{aligned} \quad (197)$$

and

$$k_{\perp}^i(s) \equiv k_{\perp}^i + i s \varepsilon_{ij} k_{\perp}^j. \quad (198)$$

Finally we result

$$P^{+2} M_{2ii}^{(ex)} = - \left\{ \delta_{s_1 s_3} \delta_{s_2 s_4} T_1^{\perp} \cdot T_2^{\perp} - \delta_{s_1 \bar{s}_2} \delta_{s_1 \bar{s}_3} \delta_{s_2 \bar{s}_4} 2m^2 \frac{(x-x')^2}{xx'(1-x)(1-x')} \right. \\ \left. + im\sqrt{2}(x'-x) \left[ \delta_{s_1 \bar{s}_3} \delta_{s_2 s_4} \frac{s_1}{xx'} T_1^{\perp} \cdot \varepsilon_{s_1}^{\perp} + \delta_{s_1 s_3} \delta_{s_2 \bar{s}_4} \frac{s_2}{(1-x)(1-x')} T_2^{\perp} \cdot \varepsilon_{s_2}^{\perp} \right] \right\}. \quad (199)$$

Whereas in the **annihilation channel** we calculate

$$P^+ [\chi_{s_3}^+ \Gamma^i(-p_4, p_3, -(p_1 + p_2)) \chi_{\bar{s}_4}] = \chi_{s_3}^+ \left[ -\frac{\sigma \cdot k'_{\perp}}{x'} \sigma^i + \sigma^i \frac{\sigma \cdot k'_{\perp}}{1-x'} + im \frac{1}{x'(1-x')} \sigma^i \right] \chi_{\bar{s}_4} \\ = T_3^i \delta_{s_3 \bar{s}_4} + im \frac{1}{x'(1-x')} (-\sqrt{2}) s_4 \varepsilon_{s_4}^{i*} \delta_{s_3 s_4} \quad (200)$$

and

$$P^+ [\chi_{\bar{s}_2}^+ \Gamma^i(p_1, -p_2, p_1 + p_2) \chi_{s_1}] = \chi_{\bar{s}_2}^+ \left[ \frac{\sigma \cdot k_{\perp}}{1-x} \sigma^i - \sigma^i \frac{\sigma \cdot k_{\perp}}{x} - im \frac{1}{x(1-x)} \sigma^i \right] \chi_{s_1} \\ = T_4^i \delta_{s_1 \bar{s}_2} - im \frac{1}{x(1-x)} (-\sqrt{2}) s_1 \varepsilon_{s_1}^i \delta_{s_1 s_2}, \quad (201)$$

where we have introduced

$$T_3^i \equiv -\frac{k'_{\perp}{}^i(\bar{s}_3)}{x'} + \frac{k'_{\perp}{}^i(s_3)}{1-x'} \\ T_4^i \equiv \frac{k_{\perp}^i(\bar{s}_1)}{1-x} - \frac{k_{\perp}^i(s_1)}{x}. \quad (202)$$

We finally have

$$P^{+2} M_{2ii}^{(an)} = \delta_{s_1 \bar{s}_2} \delta_{s_3 \bar{s}_4} T_3^{\perp} \cdot T_4^{\perp} + \delta_{s_1 s_2} \delta_{s_3 s_4} \delta_{s_1 s_3} 2m^2 \frac{1}{xx'(1-x)(1-x')} \\ + im\sqrt{2} \left[ \delta_{s_3 \bar{s}_4} \delta_{s_1 s_2} \frac{s_1}{x(1-x)} T_3^{\perp} \cdot \varepsilon_{s_1}^{\perp} - \delta_{s_3 s_4} \delta_{s_1 \bar{s}_2} \frac{s_3}{x'(1-x')} T_4^{\perp} \cdot \varepsilon_{s_4}^{\perp*} \right]. \quad (203)$$

## C Generated interaction in comparison with the result derived in the renormalization scheme of Glazek and Wilson

We derive here the generated interaction to the order  $O(e^2)$  in the renormalization scheme of Glazek and Wilson and compare it with the one derived in the scheme of Wegner. To this end we use the flow equations of type **II.**, see Appendix A, with, for sake of simplicity, the  $f$ -function chosen as

$$f_{p_i p_f, \lambda} = \theta(\lambda - |\Delta_{p_i p_f}|), \quad (204)$$

where  $\Delta_{p_i p_f} = \sum p_i^- - \sum p_f^-$ . This gives for the second order generated interaction in  $|e\bar{e}\rangle$ -sector

$$\frac{dV_{p_i p_f, \lambda}^{gen}}{d\lambda} = f_{p_i p_f, \lambda} < [\eta_{\lambda}^{(1)}, H_{\lambda}^{ee\gamma}] >_{|e\bar{e}\rangle} + \frac{df_{p_i p_f, \lambda}}{d\lambda} \frac{V_{p_i p_f, \lambda}^{eff}}{f_{p_i p_f, \lambda}} \quad (205)$$

with the solution

$$V_{p_i p_f, \lambda} = -f_{p_i p_f, \lambda} \int_{\lambda}^{\infty} d\lambda' < [\eta_{\lambda'}^{(1)}, H_{\lambda'}^{ee\gamma}] >_{|e\bar{e}\rangle}, \quad (206)$$

where the initial condition  $V_{p_i p_f}(\lambda \rightarrow \infty) = 0$  is implied. We make use of the expression for the matrix element of the commutator  $[\eta, H^{ee\gamma}]$  in the exchange and annihilation channels, eqs. (190) and (191) in Appendix B, where now the derivative is performed with respect of the UV cutoff, i.e.  $d/d\lambda$ . This results for the generated interaction in both channels

$$V_{\lambda}^{(ex)} = -e^2 M_{2ii, \lambda}^{(ex)} \frac{1}{p_1^+ - p_3^+} f_{p_i p_f, \lambda} \left[ \frac{\int_{\lambda}^{\infty} \frac{df_{p_1, p_3, \lambda'}}{d\lambda'} f_{p_4, p_2, \lambda'} d\lambda'}{\Delta_{p_1, p_3, \lambda}} + \frac{\int_{\lambda}^{\infty} \frac{df_{p_4, p_2, \lambda'}}{d\lambda'} f_{p_1, p_3, \lambda'} d\lambda'}{\Delta_{p_4, p_2, \lambda}} \right] \quad (207)$$

$$V_{\lambda}^{(an)} = e^2 M_{2ii, \lambda}^{(an)} \frac{1}{p_1^+ + p_2^+} f_{p_i p_f, \lambda} \left[ \frac{\int_{\lambda}^{\infty} \frac{df_{p_1, -p_2, \lambda'}}{d\lambda'} f_{p_4, -p_3, \lambda'} d\lambda'}{\Delta_{p_1, -p_2, \lambda}} + \frac{\int_{\lambda}^{\infty} \frac{df_{p_4, -p_3, \lambda'}}{d\lambda'} f_{p_1, -p_2, \lambda'} d\lambda'}{\Delta_{p_4, -p_3, \lambda}} \right]$$

and with  $f_{p_1 p_2}$  being according to eqs. (176) and (177)

$$V_{\lambda}^{(ex)} = -e^2 M_{2ii, \lambda}^{(ex)} \frac{1}{p_1^+ - p_3^+} \theta(\lambda - |\Delta_{p_i p_f, \lambda}|) \times \left[ \frac{\theta(|\Delta_{p_1 p_3}| - |\Delta_{p_4 p_2}|) \theta(|\Delta_{p_1 p_3}| - \lambda)}{\Delta_{p_1 p_3}} + \frac{\theta(|\Delta_{p_4 p_2}| - |\Delta_{p_1 p_3}|) \theta(|\Delta_{p_4 p_2}| - \lambda)}{\Delta_{p_4 p_2}} \right] \quad (208)$$

$$V_{\lambda}^{(an)} = e^2 M_{2ii, \lambda}^{(an)} \frac{1}{p_1^+ + p_2^+} \theta(\lambda - |\Delta_{p_i p_f, \lambda}|) \times \left[ \frac{\theta(|\Delta_{p_1, -p_2}| - |\Delta_{p_4, -p_3}|) \theta(|\Delta_{p_1, -p_2}| - \lambda)}{\Delta_{p_1, -p_2}} + \frac{\theta(|\Delta_{p_4, -p_3}| - |\Delta_{p_1, -p_2}|) \theta(|\Delta_{p_4, -p_3}| - \lambda)}{\Delta_{p_4, -p_3}} \right].$$

This result for the generated  $e\bar{e}$ -interaction was obtained by the authors of [6]. It is to be compared with the expression in the main text for the generated interaction, eq. (50). First, due to the  $\theta$ -functions the interaction in eq. (208) is also free of divergencies coming from small energy denominators. As compared with the result of flow equations, eq. (50), each energy denominator in eq. (208) has its own relative weight in the scheme of similarity transformation. We rewrite eq. (208) as the sum of two terms, the first corresponding to the result of flow equations, eq. (208), and a second term representing the rest, which carries different weights for each energy denominator.

$$V_{\lambda}^{(ex)} = -e^2 M_{2ii, \lambda}^{(ex)} \frac{1}{p_1^+ - p_3^+} \theta(\lambda - |\Delta_{p_i p_f, \lambda}|) \times \left\{ \frac{1}{2} \left( \frac{1}{\Delta_{p_1, p_3}} + \frac{1}{\Delta_{p_4, p_2}} \right) \left[ 1 - \theta(\lambda - |\Delta_{p_1, p_3}|) \theta(\lambda - |\Delta_{p_4, p_2}|) \right] \right. \\ \left. + \frac{1}{2} \left( \frac{1}{\Delta_{p_1, p_3}} - \frac{1}{\Delta_{p_4, p_2}} \right) \left[ \theta(|\Delta_{p_1, p_3}| - |\Delta_{p_4, p_2}|) \theta(|\Delta_{p_1, p_3}| - \lambda) \right] \right\}$$

$$\begin{aligned}
& -\theta(|\Delta_{p_4,p_2}| - |\Delta_{p_1,p_3}|) \theta(|\Delta_{p_4,p_2}| - \lambda) \Big] \Big\} \\
V_\lambda^{(an)} &= e^2 M_{2ii,\lambda}^{(an)} \frac{1}{p_1^+ + p_2^+} \theta(\lambda - |\Delta_{p_i p_f, \lambda}|) \\
& \times \left\{ \frac{1}{2} \left( \frac{1}{\Delta_{p_1, -p_2}} + \frac{1}{\Delta_{p_4, -p_3}} \right) \left[ 1 - \theta(\lambda - |\Delta_{p_1, -p_2}|) \theta(\lambda - |\Delta_{p_4, -p_3}|) \right] \right. \\
& + \frac{1}{2} \left( \frac{1}{\Delta_{p_1, -p_2}} - \frac{1}{\Delta_{p_4, -p_3}} \right) \left[ \theta(|\Delta_{p_1, -p_2}| - |\Delta_{p_4, -p_3}|) \theta(|\Delta_{p_1, -p_2}| - \lambda) \right. \\
& \left. \left. - \theta(|\Delta_{p_4, -p_3}| - |\Delta_{p_1, -p_2}|) \theta(|\Delta_{p_4, -p_3}| - \lambda) \right] \right\}. \quad (209)
\end{aligned}$$

Here we have used the identity

$$\frac{\theta_1}{\Delta_1} + \frac{\theta_2}{\Delta_2} = \frac{1}{2}(\theta_1 + \theta_2) \left( \frac{1}{\Delta_1} + \frac{1}{\Delta_2} \right) + \frac{1}{2}(\theta_1 - \theta_2) \left( \frac{1}{\Delta_1} - \frac{1}{\Delta_2} \right), \quad (210)$$

where in the exchange channel

$$\begin{aligned}
\theta_1 &= \int_\lambda^\infty \frac{df_{p_1, p_3, \lambda'}}{d\lambda'} f_{p_4, p_2, \lambda'} d\lambda' \\
\theta_2 &= \int_\lambda^\infty \frac{df_{p_1, -p_2, \lambda'}}{d\lambda'} f_{p_4, -p_3, \lambda'} d\lambda'
\end{aligned} \quad (211)$$

and

$$\theta_1 + \theta_2 = 1 - f_{p_1 p_3} f_{p_4 p_2}. \quad (212)$$

The term corresponding to the annihilation channel is treated along the same line.

For completeness, we rewrite the result produced by the flow equations of Wegner, eq. (50), as follows

$$V_{gen}^{(ex)}(l_\lambda) = -e^2 M_{2ii}^{(ex)} \frac{1}{p_1^+ - p_3^+} \frac{1}{2} \left( \frac{1}{\Delta_{p_1, p_3}} + \frac{1}{\Delta_{p_4, p_2}} \right) \left( 1 - e^{-2 \frac{\Delta_{p_1, p_3}}{\lambda} \frac{\Delta_{p_4, p_2}}{\lambda}} \right) \cdot e^{-\left( \frac{\Delta_{p_i p_f}}{\lambda} \right)^2} \quad (213)$$

$$V_{gen}^{(an)}(l_\lambda) = e^2 M_{2ii}^{(an)} \frac{1}{p_1^+ + p_2^+} \frac{1}{2} \left( \frac{1}{\Delta_{p_1, -p_2}} + \frac{1}{\Delta_{p_4, -p_3}} \right) \left( 1 - e^{-2 \frac{\Delta_{p_1, -p_2}}{\lambda} \frac{\Delta_{p_4, -p_3}}{\lambda}} \right) \cdot e^{-\left( \frac{\Delta_{p_i p_f}}{\lambda} \right)^2}.$$

This interaction corresponds to the first term in eq. (208). In the nonrelativistic approximation the second term with the difference of energy denominators vanishes; hence it does not contribute to the spectrum of positronium (see main text). In general it is an open question, whether the second term contributes to the physical values. This would mean the two methods not to be equivalent in general.

## D Fermion and photon self energy terms

We calculate here the fermion and photon self energy terms, arising from the second order commutator  $[\eta^{(1)}, H_{ee\gamma}]$ .

**I.** We first derive the **electron self energy** terms. Making use of the expressions for the generator of the unitary transformation  $\eta^{(1)}$  defined in eq. (37) and of  $H_{ee\gamma}$ , eq. (29), we obtain the following expression for the commutator in the electron self energy sector

$$\frac{1}{2}(\eta_{p_1 p_2} g_{p_2 p_1} - \eta_{p_2 p_1} g_{p_1 p_2}) \left[ \theta(p_1^+) \frac{\theta(p_2^+ - p_1^+)}{p_2^+ - p_1^+} \theta(p_2^+) b_{p_2}^+ b_{p_2} \chi_{s_2}^+ \chi_{s_2} \right. \\ \left. - \theta(p_2^+) \frac{\theta(p_1^+ - p_2^+)}{p_1^+ - p_2^+} \theta(p_1^+) b_{p_1}^+ b_{p_1} \chi_{s_1}^+ \chi_{s_1} \right] M_{2ij}(p_1, p_2) \delta^{ij}, \quad (214)$$

where

$$M_{2ij}(p_1, p_2) = \Gamma^i(p_1, p_2, p_1 - p_2) \Gamma^j(p_2, p_1, p_2 - p_1) \quad (215)$$

and the momentum integration over  $p_1, p_2$  is implied;  $1/2$  stands as the symmetry factor. The matrix element of the commutator between the free fermion states is

$$\langle p_1, s_1 | [\eta^{(1)}, H_{ee\gamma}] | p_1, s_1 \rangle_{selfenergy} \\ = - \int_{p_2} (\eta_{p_1 p_2} g_{p_2 p_1} - \eta_{p_2 p_1} g_{p_1 p_2}) \theta(p_2^+) \frac{\theta(p_1^+ - p_2^+)}{p_1^+ - p_2^+} M_{2ii}(p_1, p_2), \quad (216)$$

where the integration  $\int_p$  is defined in eq. (34). We use the expression for the generator  $\eta$  through the coupling, namely

$$\eta_{p_1 p_2} g_{p_2 p_1} - \eta_{p_2 p_1} g_{p_1 p_2} = \frac{1}{\Delta_{p_1 p_2}} \left( g_{p_1 p_2} \frac{dg_{p_2 p_1}}{dl} + g_{p_2 p_1} \frac{dg_{p_1 p_2}}{dl} \right). \quad (217)$$

Change of the variables according to

$$\begin{aligned} p_1 &= p \\ p_2 &= p_k \\ p_1 - p_2 &= k \end{aligned} \quad (218)$$

brings the integral in eq. (216) to the standard form of loop integration

$$- \int_k (\eta_{p, p-k} g_{p-k, p} - \eta_{p-k, p} g_{p, p-k}) \theta(p^+ - k^+) \frac{\theta(k^+)}{k^+} M_{2ii}(p, p - k). \quad (219)$$

According to eq. (57), the integral  $\int_{l_\lambda}^{l_\Lambda}$  of the commutator  $[\eta^{(1)}, H_{ee\gamma}]$  defines the difference between the energies (or energy corrections)  $\delta p_{1\lambda}^- - \delta p_{1\Lambda}^-$ . Making use of

$$\int_{l_\Lambda}^{l_\lambda} dl' (\eta_{p_1 p_2} g_{p_2 p_1} - \eta_{p_2 p_1} g_{p_1 p_2}) = \frac{1}{p_1^- - p_2^- - (p_1 - p_2)^-} (g_{p_1, p_2, \Lambda} g_{p_2, p_1, \lambda} - g_{p_1, p_2, \lambda} g_{p_2, p_1, \Lambda}) \quad (220)$$

we have the following explicit expression:

$$\delta p_{1\lambda}^- - \delta p_{1\Lambda}^- = e^2 \int \frac{d^2 k^\perp dk^+}{2(2\pi)^3} \frac{\theta(k^+)}{k^+} \theta(p^+ - k^+) \frac{(-1)}{p^- - k^- - (p - k)^-} \quad (221) \\ \times \Gamma^i(p - k, p, -k) \Gamma^i(p, p - k, k) \left[ \exp \left\{ -2 \left( \frac{\Delta_{p, p-k}}{\lambda} \right)^2 \right\} - \exp \left\{ -2 \left( \frac{\Delta_{p, p-k}}{\Lambda} \right)^2 \right\} \right],$$

where the solution for the  $ee\gamma$ -coupling constant was used. Therefore the electron energy correction corresponding to the first diagram, cf. fig. (2), is

$$\delta p_{1\lambda}^- = e^2 \int \frac{d^2 k^\perp dk^+}{2(2\pi)^3} \frac{\theta(k^+)}{k^+} \theta(p^+ - k^+) \quad (222) \\ \times \Gamma^i(p - k, p, -k) \Gamma^i(p, p - k, k) \frac{1}{p^- - k^- - (p - k)^-} \times (-R),$$

where we have introduced the regulator  $R$ , defining the cutoff condition (see main text),

$$R = \exp \left\{ -2 \left( \frac{\Delta_{p,k}}{\lambda} \right)^2 \right\} \quad (223)$$

(note that  $\Delta_{p,k} = \Delta_{p,p-k}$ ). To perform the integration over  $k = (k^+, k^\perp)$  explicitly, choose the parametrization

$$\begin{aligned} \frac{k^+}{p^+} &= x \\ k &= (xp^+, xp^\perp + \kappa^\perp), \end{aligned} \quad (224)$$

where  $p = (p^+, p^\perp)$  is the external electron momentum. Then the terms occuring in  $\delta p_{1\lambda}^-$  are rewritten in the form

$$\begin{aligned} \Gamma^i(p-k, p, -k) \Gamma^i(p, p-k, k) &= \frac{1}{(p^+)^2(1-x)^2} \left( \left( 4\frac{1}{x^2} - 4\frac{1}{x} + 2 \right) \kappa_\perp^2 + 2m^2 x^2 \right) \\ \Delta_{p,p-k} = p^- - k^- - (p-k)^- &= \frac{1}{p^+x(1-x)} (x(1-x)p^2 - \kappa_\perp^2 - xm^2) = \frac{\tilde{\Delta}_{p,p-k}}{p^+} \quad . \end{aligned} \quad (225)$$

Therefore the integral for the electron energy correction corresponding to the first diagram of fig. (2) takes the form

$$\begin{aligned} p^+ \delta p_{1\lambda}^- &= -\frac{e^2}{8\pi^2} \int_0^1 dx \int d\kappa_\perp^2 \frac{\left( \frac{2}{x^2} - \frac{2}{x} + 1 \right) \kappa_\perp^2 + m^2 x^2}{(1-x)(\kappa_\perp^2 + f(x))} \times (-R) \\ &= -\frac{e^2}{8\pi^2} \int_0^1 dx \int d\kappa_\perp^2 \\ &\quad \times \left[ \frac{p^2 - m^2}{\kappa_\perp^2 + f(x)} \left( \frac{2}{[x]} - 2 + x \right) - \frac{2m^2}{\kappa_\perp^2 + f(x)} + \left( \frac{2}{[x]^2} + \frac{1}{[1-x]} \right) \right] \times (-R), \end{aligned} \quad (226)$$

where

$$f(x) = xm^2 - x(1-x)p^2. \quad (227)$$

In the last integral the principal value prescription for  $\frac{1}{[x]}$  as  $x \rightarrow 0$  was introduced (see main text), to regularize the IR divergencies present in the longitudinal direction.

We thus have derived the expression for the energy correction which has been used in the main text.

**II.** We repeat the same procedure for the **photon self energy**. The second order commutator  $[\eta^{(1)}, H_{ee\gamma}]$  gives the following expression in the photon self energy sector

$$\begin{aligned} \frac{1}{2} (\eta_{p_1 p_2} g_{p_2 p_1} - \eta_{p_2 p_1} g_{p_1 p_2}) \cdot \left[ \theta(p_1^+) \theta(-p_2^+) \frac{\theta(p_1^+ - p_2^+)}{(p_1^+ - p_2^+)} a_{-q}^+ a_{-q} \varepsilon_\lambda^{i*} \varepsilon_\lambda^j \right. \\ \left. - \theta(-p_1^+) \theta(p_2^+) \frac{\theta(p_2^+ - p_1^+)}{(p_2^+ - p_1^+)} a_q^+ a_q \varepsilon_\lambda^i \varepsilon_\lambda^{j*} \right] \cdot Tr M_{2ij}(p_1, p_2) \delta_{q, -(p_1 - p_2)}, \end{aligned} \quad (228)$$

where  $M_{2ij}(p_1, p_2)$  is defined in eq. (215) and the trace acts in spin space; the integration over the momenta  $q, p_1$  and  $p_2$  is implied. The matrix element between the free photon states reads

$$\begin{aligned} \langle q, \lambda | [\eta^{(1)}, H_{ee\gamma}] | q, \lambda \rangle_{selfenergy} \delta_{ij} \\ = -\frac{1}{q^+} \int_{p_1, p_2} (\eta_{p_1 p_2} g_{p_2 p_1} - \eta_{p_2 p_1} g_{p_1 p_2}) \theta(-p_1^+) \theta(p_2^+) Tr M_{2ij}(p_1, p_2) \delta_{q, -(p_1 - p_2)}, \end{aligned} \quad (229)$$



that can be rewritten after the change of coordinates according to

$$\begin{aligned} p_1 &= -k \\ p_2 &= -(k - q) \\ p_2 - p_1 &= q \end{aligned} \quad (230)$$

in the following way

$$\frac{1}{q^+} \int_k (\eta_{k,k-q} g_{k-q,k} - \eta_{k-q,k} g_{k,k-q}) \theta(k^+) \theta(q^+ - k^+) \text{Tr} M_{2ij}(k, k - q), \quad (231)$$

where the symmetry

$$\begin{aligned} \eta_{-p_1, -p_2} &= -\eta_{p_1, p_2} \\ g_{-p_1, -p_2} &= g_{p_1, p_2} \end{aligned} \quad (232)$$

has been used. The integration of the commutator over  $l$  in the flow equation gives rise to

$$\begin{aligned} (\delta q_{1\lambda}^- - \delta q_{1\Lambda}^-) \delta^{ij} &= \frac{1}{q^+} e^2 \int \frac{d^2 k^\perp dk^+}{2(2\pi)^3} \theta(k^+) \theta(q^+ - k^+) \frac{(-1)}{q^- - k^- - (q - k)^-} \\ &\times \text{Tr} \left( \Gamma^i(k, k - q, q) \Gamma^j(k - q, k, -q) \right) \left[ \exp \left\{ -2 \left( \frac{\Delta_{q,q-k}}{\lambda} \right)^2 \right\} - \exp \left\{ -2 \left( \frac{\Delta_{q,q-k}}{\Lambda} \right)^2 \right\} \right]. \end{aligned} \quad (233)$$

This means for the photon energy correction

$$\begin{aligned} \delta q_{1\lambda}^- \delta^{ij} &= \frac{1}{q^+} e^2 \int \frac{d^2 k^\perp dk^+}{2(2\pi)^3} \theta(k^+) \theta(q^+ - k^+) \\ &\times \text{Tr} \left( \Gamma^i(k, k - q, q) \Gamma^j(k - q, k, -q) \right) \frac{1}{q^- - k^- - (q - k)^-} \times (-R), \end{aligned} \quad (234)$$

where the regulator  $R$

$$R = \exp \left\{ -2 \left( \frac{\Delta_{q,k}}{\lambda} \right)^2 \right\} \quad (235)$$

has been introduced. Define the new set of coordinates

$$\begin{aligned} \frac{(q - k)^+}{q^+} &= x \\ k &= ((1 - x)q^+, (1 - x)q^\perp + \kappa^\perp) \\ q - k &= (xq^+, xq^\perp - \kappa^\perp), \end{aligned} \quad (236)$$

where  $q = (q^+, q^\perp)$  is the photon momentum. Then the terms present in  $\delta q_{1\lambda}^-$  are

$$\begin{aligned} \Gamma^i(k, k - q, q) \Gamma^i(k - q, k, -q) &= \frac{2}{(q^+)^2 x (1-x)^2} \left( \left( 2x - 2 + \frac{1}{x} \right) \kappa_\perp^2 + \frac{m^2}{x} \right) \\ \Delta_{k-q,k} &= q^- - k^- - (q - k)^- = -\frac{\kappa_\perp^2 + m^2}{q^+ x (1-x)} + \frac{q^2}{q^+} = \frac{\tilde{\Delta}_{k-q,k}}{q^+}. \end{aligned} \quad (237)$$

The integral for the photon energy correction corresponding to the first diagram of fig. (3) takes the form

$$\begin{aligned} q^+ \delta q_{1\lambda}^- &= -\frac{e^2}{8\pi^2} \int_0^1 dx \int d\kappa_\perp^2 \frac{(2x - 2 + \frac{1}{x}) \kappa_\perp^2 + \frac{m^2}{x}}{(1-x)(\kappa_\perp^2 + f(x))} \times (-R) \\ &= -\frac{e^2}{8\pi^2} \int_0^1 dx \int d\kappa_\perp^2 \left\{ \frac{q^2}{\kappa_\perp^2 + f(x)} (2x^2 - 2x + 1) + \frac{2m^2}{[1-x]} + \left( -2 + \frac{1}{[x][1-x]} \right) \right\} \times (-R) \end{aligned} \quad (238)$$

with

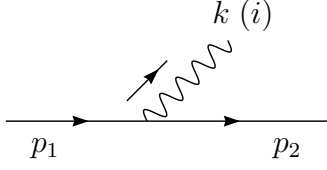
$$f(x) = m^2 - q^2 x(1 - x) , \quad (239)$$

and the principal value prescription, denoted by '[ ]', introduced to regularize the IR divergencies.

This is the form of the photon correction used in the main text.

## References

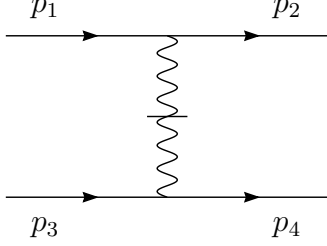
- [1] F. Wegner, Ann.Physik **3**,77 (1994).
- [2] S. D. Glazek and K. G. Wilson, Phys. Rev. **D48**, 5863 (1993); S. D. Glazek and K. G. Wilson, Phys.Rev. **D49**, 4214 (1994).
- [3] A. Mielke, Lectures "Flußgleichungen für Hamiltonoperatoren", held at the Institut für theoretische Physik der Universität Heidelberg.
- [4] R. G. Perry and K. G. Wilson, Nucl.Phys. **B403**, 587 (1993).
- [5] W. M. Zhang and A. Harindranath, Phys.Rev. **D48** 4868 (1993); *ibid.* 4881; *ibid.* 4903.
- [6] B. D. Jones, R. G. Perry and S. D. Glazek, hep-th/9605231.
- [7] J. Müller and J. Rau, Phys.Lett. **B386**, 274 (1996).



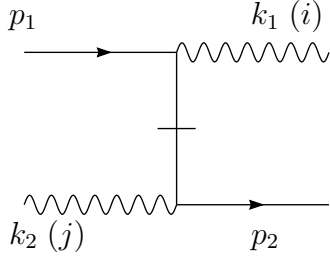
$$-e_\lambda f_{p_i p_f, \lambda} \cdot \chi_2^+ \Gamma_\lambda^i(p_1, p_2, k) \chi_1 \varepsilon^{i*}$$

$$\Gamma_\lambda^i(p_1, p_2, k) = 2 \frac{k^i}{[k^+]} - \frac{\sigma \cdot p_2^\perp - i m_\lambda}{[p_2^+]} \sigma^i - \sigma^i \frac{\sigma \cdot p_1^\perp + i m_\lambda}{[p_1^+]}$$

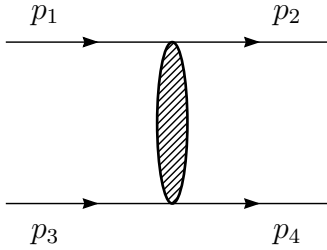
$$i = 1, 2$$



$$e_\lambda^2 f_{p_i p_f, \lambda} \cdot \chi_3^+ \chi_4^+ \frac{4}{[p_1^+ - p_2^+]^2} \chi_1 \chi_2$$



$$e_\lambda^2 f_{p_i p_f, \lambda} \cdot \chi_2^+ \frac{\sigma^j \sigma^i}{[p_1^+ - k_1^+]} \chi_1 \varepsilon^{i*} \varepsilon^j$$

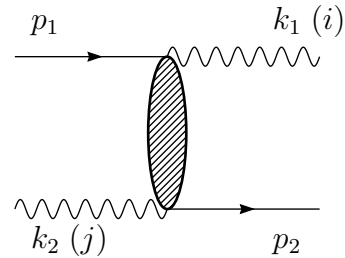


$$e_\lambda^2 f_{p_i p_f, \lambda} \cdot M_{2ij, \lambda} \delta^{ij} \cdot \frac{1}{[p_1^+ - p_2^+]}$$

$$\times \frac{1}{2} \left( \frac{1}{\Delta_{p_1 p_2 \lambda}} + \frac{1}{\Delta_{p_4 p_3 \lambda}} \right) \left( 1 - \exp \left\{ -2 \frac{\Delta_{p_1 p_2 \lambda} \cdot \Delta_{p_4 p_3 \lambda}}{\lambda^2} \right\} \right)$$

$$M_{2ij, \lambda} = \left( \chi_2^+ \Gamma_\lambda^i(p_1, p_2, p_1 - p_2) \chi_1 \right)$$

$$\times \left( \chi_4^+ \Gamma_\lambda^j(p_3, p_4, -(p_1 - p_2)) \chi_3 \right)$$



$$e_\lambda^2 f_{p_i p_f, \lambda} \cdot \widetilde{M}_{2ij, \lambda} \varepsilon^{i*} \varepsilon^j$$

$$\times \frac{1}{2} \left( \frac{1}{\Delta_{p_1 k_1 \lambda}} + \frac{1}{\Delta_{p_2 k_2 \lambda}} \right) \left( 1 - \exp \left\{ -2 \frac{\Delta_{p_1 k_1 \lambda} \cdot \Delta_{p_2 k_2 \lambda}}{\lambda^2} \right\} \right)$$

$$\widetilde{M}_{2ij, \lambda} = \chi_2^+ \Gamma_\lambda^i(p_1, p_1 - k_1, k_1) \Gamma_\lambda^j(p_1 - k_1, p_2, k_2) \chi_1$$

Figure 1: Renormalized to the second order  $O(e^2)$  light cone theory (the UV cutoff is  $\lambda$ ). The photon momenta are  $x^+$ -ordered, from left to right. The similarity function  $f_{p_i p_f, \lambda} = \exp(-\Delta_{p_i p_f, \lambda}^2 / \lambda^2)$  plays the role of regulator, where  $\Delta_{p_i p_f, \lambda} = \Sigma p_i^- - \Sigma p_f^-$  (the index ‘ $i$ ’ denotes initial and ‘ $f$ ’ final states) and  $\Delta_{p_1 p_2 \lambda} = p_1^- - p_2^- - (p_1 - p_2)^-$ ,  $p^- = (p_\perp^2 + m_\lambda^2) / p^+$ .

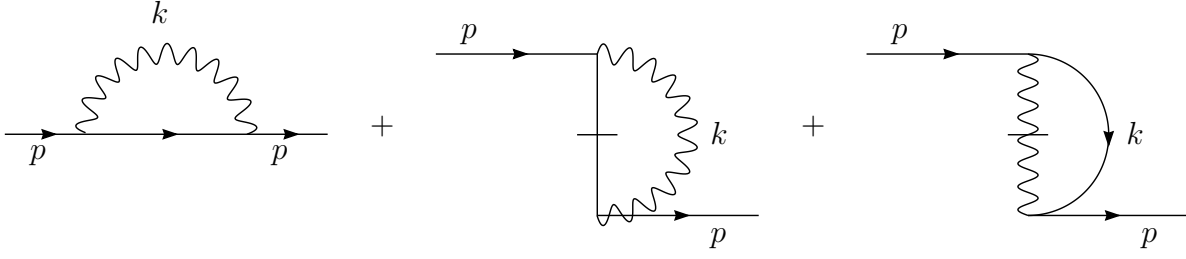


Figure 2: Electron self energy: the first diagram corresponds to the commutator term  $[\eta^{(1)}, H_{ee\gamma}]$  in the electron self energy sector, next two diagrams arise from the normal ordering of instantaneous interactions.

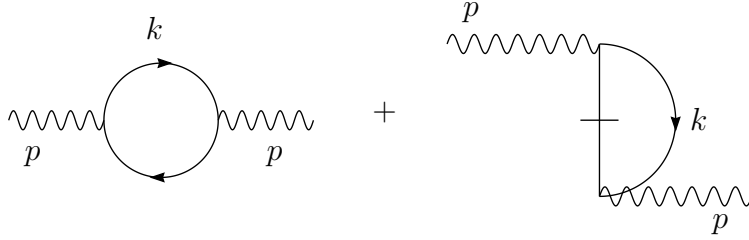


Figure 3: Photon self energy: the first diagram comes from the commutator  $[\eta^{(1)}, H_{ee\gamma}]$  in the photon self energy sector, the second one from the normal ordering of the instantaneous interaction.

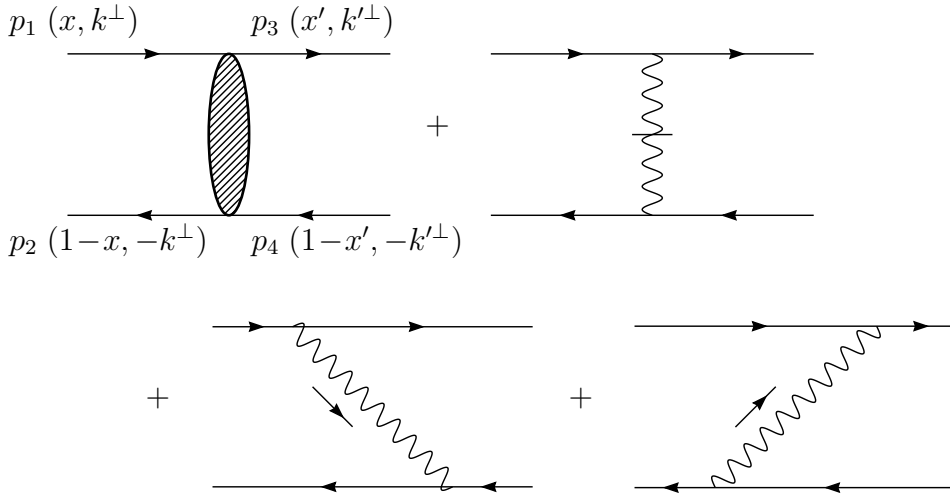


Figure 4: The renormalized to the second order electron-positron interaction in the exchange channel; diagrams correspond to generated, instantaneous interactions and two perturbative photon exchanges with respect to different time ordering.